Fourier Restriction Norm Method

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Abstract

The Fourier restriction norm method ($X^{s,b}$ spaces) introduced by Bourgain in 1993 has been highly influential in the study of dispersive partial differential equations and harmonic analysis. This method mainly takes advantage of the space-time frequency localisation of solutions to dispersive partial differential equations on the torus. Through this, multilinear estimates can be reduced to discrete counting problems, allowing well-posedness results to be obtained (existence, uniqueness, stability under perturbation of solutions).

We study the local-in-time well-posedness problem for the cubic nonlinear Schrödinger equation on the torus, in one and two dimensions. We first establish the general methodology in the Euclidean setting, which we then adapt to the torus following methods introduced in Bourgain's seminal 1993 work. We also remark on the literature and recent well-posedness results for the cubic nonlinear Schrödinger equation.

Declaration

I declare that this thesis was composed by myself, that the work contained therein is my own, except where explicitly stated otherwise in the text, and that no Generative AI was used in the completion of this work.

(*Halit Yiğit Gündüz*)

To everyone who thought and felt with me

Contents

Abstract ii

Contents v

1 Introduction 1

2 Preliminaries 3
- 2.1 Dyadic decompositions . . . 6
- 2.2 The Duhamel formulation . . . 7
- 2.3 Perturbation . . . 9
- 2.4 Contraction mapping arguments . . . 10

3 Well-posedness of the nonlinear Schrödinger equation for $s > \frac{d}{2}$ 11

4 Well-posedness of the nonlinear Schrödinger equation on $\mathbb{R}^d$ 15
- 4.1 Scaling heuristics . . . 15
- 4.2 Strichartz estimates . . . 17

5 Fourier restriction norm spaces 24
- 5.1 Linear estimates . . . 29
- 5.2 Transference principle . . . 36
- 5.3 An alternative definition . . . 38

6 Cubic nonlinear Schrödinger equation on $\mathbb{T}$ 39
- 6.1 Strichartz estimate . . . 39
- 6.2 Contraction mapping argument . . . 44

7 Cubic nonlinear Schrödinger equation on $\mathbb{T}^2$ 46
- 7.1 Strichartz estimate . . . 46
- 7.2 Contraction mapping argument . . . 51

8 Conclusion 58

Chapter 1

Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u) \\ u\big|_{t=0} = u_0, \end{cases} \qquad u : \mathbb{R} \times M^d \to \mathbb{C}, \tag{NLS}$$

where $M = \mathbb{R}$ or $M = \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, and

$$\mathcal{N}(u) := \pm|u|^{p-1}u$$

with $p \in \mathbb{N}$ an odd number. A negative nonlinearity enforces a focusing regime, and a positive one enforces a defocusing regime, although for the majority of our discussion we will not make such a distinction.

The Cauchy problem (NLS) appears in many areas of physics, including the study of nonlinear optics and fluid dynamics. It is part of a larger class of dispersive partial differential equations, which includes the Korteweg–de Vries (KdV) equations and several variants thereof. Dispersive equations have been extensively studied, both from an applied and a theoretical point of view. The main problem we consider is of *well-posedness*. We wish to show that solutions to the Cauchy problem (NLS):

- exist for given initial data $u_0 \in X$,
- are unique for any $u_0 \in X$, and that
- the solution map $u_0 \mapsto u$ is continuous with respect to initial data, or in other words, the solutions have stability under perturbation to initial data,

where X is some function space. If the above conditions are satisfied for some local time interval $t \in [0, T]$ where $0 < T < 1$, then we say that (NLS) is *locally well-posed* in X, and if these conditions hold for all $t \in \mathbb{R}$, then we say that (NLS) is *globally well-posed* in X. If well-posedness does not hold, then we say that (NLS) is *ill-posed*.

Multilinear harmonic analysis has been fundamental to the theoretical understanding of dispersive equations. There is a body of work by Kenig–Ponce–Vega developing this methodology in the 1980s and the 1990s, mainly on KdV-type

equations, see for instance [18, 19, 20, 21]. These methods are mainly applicable in the Euclidean setting ($x \in \mathbb{R}^d$). A major breakthrough in the study of dispersive equations came around this time with the introduction of the *Fourier restriction norm method* by Bourgain in 1993 [3]. In this paper, many well-posedness results were established for both (NLS) and KdV-type equations through multilinear estimates in the periodic setting ($x \in \mathbb{T}^d$). We will be considering some of these results concerning (NLS). It is worth noting that the modern form of Fourier restriction norm spaces introduced in [3] had precursors in [1, 23, 26].

The Fourier restriction norm method has been widely influential, leading to many well-posedness and ill-posedness results through similar methodology [6, 12] (see Chapter 8 for a more detailed discussion). See also [31] for a systematic study of multilinear estimates using Fourier restriction norm spaces for dispersive equations. The method also has connections to some of the major problems in harmonic analysis, primarily in restriction theory and maximal functions [24, 29, 34], see also [9].

We wish to present some of the major results in [3]. We will work exclusively with the nonlinear Schrödinger equation (NLS) on the torus $\mathbb{T}^d$ where $d = 1, 2$, with a focus on the cubic nonlinearity ($p = 3$). However, most of the methodology we will employ is applicable to other (dispersive) partial differential equations, such as KdV-type equations, and a general discussion of these methods can be found in, for instance [32]. In particular, we will prove the theorems:

Theorem 1.0.1. *The cubic nonlinear Schrödinger equation is locally well-posed in* $L^2(\mathbb{T})$.

Theorem 1.0.2. *The cubic nonlinear Schrödinger equation is locally well-posed in* $H^s(\mathbb{T}^2)$ *for* $s > 0$.

We will prove Theorem 1.0.1 in Chapter 6 and Theorem 1.0.2 in Chapter 7. Leading up to this, we will discuss preliminaries in Chapter 2, prove local well-posedness in the high regularity setting in Chapter 3, and establish the framework of Strichartz estimates along with some well-posedness proofs on $\mathbb{R}^d$ in Chapter 4.

Chapter 2

Preliminaries

We begin this chapter by introducing some notation. As we will not mind constants in inequalities, we introduce the following:

- We will write $X \lesssim Y$ to mean $X \leq cY$ for some constant $c > 0$ not dependent on X, Y. If the implicit constant is small, that is $0 < c < 1$, then we may write $X \ll Y$.
- We may index the symbol $\lesssim$ to denote that the implicit constant is dependent on some quantity or function. For instance, $\int_1^\infty x^{-p} \lesssim_p 1$ for $p > 1$.
- If $X \lesssim Y$, we may write $X = O(Y)$, in which case $Z + X = Z + O(Y)$.
- If $X \lesssim Y$ and $X \gtrsim Y$, then we write $X \sim Y$.

We are interested in analysing the NLS on the torus $\mathbb{T}^d$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We define the *Schwartz space* $\mathcal{S}(M)$ where $M = \mathbb{R}^d$ or $\mathbb{T}^d$ as the space of smooth functions $f : M \to \mathbb{C}$ such that

$$\|x^\alpha \partial_x^\beta f(x)\|_{L^\infty} < \infty, \quad \text{for all multi-indices } \alpha \in \mathbb{N}^d,\ \beta \in \mathbb{Z}^d_{\geq 0}.$$

Here, we write $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $\partial_x^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_d}^{\beta_d}$. We may similarly define $\mathcal{S}(\mathbb{Z}^d)$ as the space of functions $f : \mathbb{Z}^d \to \mathbb{C}$ such that

$$\|(1 + |n|)^\alpha f\|_{\ell^\infty} < \infty, \quad \text{for all multi-indices } \alpha \in \mathbb{N}^d.$$

If $f \in \mathcal{S}$, then we say f is *Schwartz*. When referring to function spaces, we will oftentimes index them with the relevant time or space variable.

Denoting $M = \mathbb{R}$ or $\mathbb{T}$, and $\widehat{\mathbb{R}} = \mathbb{R}$, $\widehat{\mathbb{T}} = \mathbb{Z}$, the *Fourier transform* is an operator $\mathcal{F} : \mathcal{S}(M^d) \to \mathcal{S}(\widehat{M}^d)$, defined by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^d,$$
$$\text{or} \quad \mathcal{F}(f)(n) = \widehat{f}(n) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-ix\cdot n}\, dx, \qquad n \in \mathbb{Z}^d.$$

We denote by ξ if continuous, or n if discrete, the spatial frequency variable, and by τ the temporal frequency variable. When we perform a Fourier transform in the

space and/or time variable, we will index the operator $\mathcal{F}$ accordingly or otherwise make it clear from the function's arguments. For instance, if $u : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, then

$$\mathcal{F}_x(u(t,\cdot))(n) = \widehat{u}(t,n), \qquad \mathcal{F}_{t,x}(u)(\tau,n) = \widehat{u}(\tau,n).$$

The Fourier transform is an invertible operator, with the inversion formula

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi \qquad \text{or} \qquad f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)\, e^{ix\cdot n},$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{T}^d)$, respectively. We will, unless relevant, omit the factor $\frac{1}{(2\pi)^d}$ appearing in the Fourier transform or its inverse for brevity. Formally, we are absorbing this factor into the implicit constant c_d in whichever relevant bound.

By a standard density argument, the Fourier transform can be extended to a unitary operator between the appropriate L^2 spaces. This unitary property we call Plancherel's identity. We will make use of other standard identities of the Fourier transform, such as

$$\widehat{fg} = \widehat{f} * \widehat{g}$$

for $f, g \in \mathcal{S}$, or more generally $f, g \in L^2$. For proofs of these identities and a more general discussion on the Fourier transform, see [10].

For $s \in \mathbb{R}$, we define the L^2-based nonhomogeneous Sobolev space $H^s(\mathbb{R}^d)$ as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{H^s(\mathbb{R}^d)} := \|\langle\xi\rangle^s\, \widehat{f}(\xi)\|_{L^2(\mathbb{R}^d)},$$

where $\langle\,\cdot\,\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. The definition of $H^s(\mathbb{T}^d)$ is similar. We also have the homogeneous L^2-based Sobolev spaces $\dot{H}^s$ defined in the same way, only changing from the Japanese brackets $\langle\,\cdot\,\rangle$ to the usual $|\cdot|$. We may define more general L^p-based Sobolev spaces in this way also, and we denote by $\dot{W}^{s,p}$ and $W^{s,p}$ the homogeneous and nonhomogeneous variants. It is easy to see that $H^{s_1} \subset H^{s_2}$ if and only if $s_1 \geq s_2$. We also have the Sobolev inequality, which states that

$$\|f\|_{L^q} \lesssim \|f\|_{\dot{W}^{s,p}} \tag{2.0.1}$$

for $1 < p < q < \infty$ such that $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$, where d is the dimension of the underlying space $\mathbb{R}^d$ or $\mathbb{T}^d$. The inequality above is equivalent to the following Hardy–Littlewood–Sobolev inequality:

$$\|\frac{1}{|x|^{\frac{d}{p}}} * f\|_{L^q} \lesssim \|f\|_{L^q} \tag{2.0.2}$$

for $1 < p, q, r < \infty$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, and d is as earlier. These identities and their proofs are found in [11, Chapter 6]. See also [2].

The Sobolev spaces defined above are motivated by the behaviour of derivatives

under the Fourier transform. For instance, on $\mathbb{R}$ we have

$$\mathcal{F}(\partial_x^\alpha f)(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$$

for $\alpha \in \mathbb{N}$. By viewing differential operators as Fourier multipliers, we can define fractional derivatives by allowing α to be fractional. Thus, it is natural to write

$$\|f\|_{H_x^s} = \|\langle \partial_x \rangle^s f\|_{L_x^2}.$$

So, we may heuristically view the statement $f \in H_x^s$ as $\partial_x^s f \in L_x^2$. In other words, s is some smoothness parameter. We are primarily concerned with the mixed norm space $C_T H_x^s([0,T] \times M^d) = L_t^\infty H_x^s([0,T] \times M^d)$ where $M = \mathbb{R}$ or $\mathbb{T}$, which is defined for a given $0 < T < 1$ with the norm

$$\|f\|_{C_T H_x^s} := \sup_{t \in [0,T]} \|f(t,\cdot)\|_{H_x^s}.$$

We also review some important lemmas that allow us to interpolate between inequalities:

Lemma 2.0.1 (Log-convexity of L^p norms)**.** *Let f be a measurable function, and for $1 \le q \le r \le \infty$ and $0 \le \theta \le 1$, let p be defined by*

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{r}.$$

Then,

$$\|f\|_{L^p} \le \|f\|_{L^q}^{1-\theta} \|f\|_{L^r}^{\theta}.$$

The above lemma is a direct consequence of Hölder's inequality and can be viewed as a special case of the lemma below:

Lemma 2.0.2 (Riesz–Thorin interpolation)**.** *Let (X,μ) and (Y,ν) be two σ-finite measure spaces. Let T be a linear operator defined on the set of all finitely simple functions on X and taking values in the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and assume that for all finitely simple functions f on X,*

$$\|T(f)\|_{L^{q_0}(Y)} \le M_0 \|f\|_{L^{p_0}(X)},$$
$$\|T(f)\|_{L^{q_1}(Y)} \le M_1 \|f\|_{L^{p_1}(X)}.$$

Then for all $0 < \theta < 1$, we have

$$\|T(f)\|_{L^q(Y)} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p(X)},$$

for all finitely simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Consequently, when $p < \infty$, by density, T has a unique bounded extension from $L^p(X, \mu)$ to $L^q(Y, \nu)$ with p and q as above.

The statement and proof of Lemma 2.0.2 can be found in [10, Chapter 1.3.2].

2.1 Dyadic decompositions

Dyadic decomposition is a useful tool to obtain many of the estimates we seek. Let $\varphi : \mathbb{R}_{\geq 0} \to [0, 1]$ be a smooth monotonic function such that $\varphi(r) = 1$ for $r \leq 1$ and $\varphi(r) = 0$ for $r \geq 2$. Let

- $\varphi_j(r) := \varphi(\frac{r}{2^j})$ for $j \in \mathbb{Z}_{\geq 0}$,
- $\psi_j(r) := \varphi_{j+1}(r) - \varphi_j(r)$ for $j > 0$, and
- $\psi_0(r) := \varphi(r)$.

Then, we have a partition of unity

$$\sum_{j \geq 0} \psi_j(|\xi|) = 1, \qquad \text{for all } \xi \in \mathbb{R}^d.$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ and $N := 2^j$ for some $j \in \mathbb{Z}_{\geq 0}$. We define the frequency projection operators P_N, $P_{\leq N}$ as the Fourier multipliers given by

$$\widehat{P_N f}(\xi) = \psi_j(|\xi|)\, \widehat{f}(\xi), \qquad \widehat{P_{\leq N} f}(\xi) = \varphi_j(|\xi|)\, \widehat{f}(\xi).$$

If we are on the torus, and so $f : \mathbb{T}^d \to \mathbb{C}$, then we can simply define these projectors by

$$\widehat{P_N f}(n) = \mathbb{1}_{\sim N}(|n|)\, \widehat{f}(n), \qquad \widehat{P_{\leq N} f}(n) = \mathbb{1}_{\leq N}(|n|)\, \widehat{f}(n).$$

In general, we may define frequency projections to some set $E \subset \mathbb{R}^d$ or $\mathbb{Z}^d$ by $\widehat{P_E f} = \mathbb{1}_E\, \widehat{f}$.

Remark 2.1.1. *When speaking of dyadic intervals, we will repurpose our notation and write, for instance, $|\xi| \sim N$ or $N_1 \lesssim N_2$. In this case, the notation has the usual meaning, except that the implicit constant is 2^a for $-1 \leq a < 1$, or we simply mean $|\xi| < 2$ if the dyadic integer we are comparing to is $N = 2^0 = 1$.*

These projection operators are particularly useful in situations where it is easy to show a bound when there is frequency localisation. Suppose for instance that we want to prove a bound of the form $\|f\|_{L^2} \lesssim Y$. Then, it suffices to show that $\|P_N f\|_{L^2} \lesssim N^{-\varepsilon}\, Y$ for $\varepsilon > 0$, since then we have

$$\|f\|_{L^2}^2 = \sum_{N \text{ dyadic}} \|P_N f\|_{L^2}^2 \lesssim \sum_{N \text{ dyadic}} (N^{-\varepsilon}\, Y)^2 \lesssim Y^2.$$

With arguments like these, we are employing the uncertainty principle. This is the principle that a function cannot be sharply localised in both the spatial (or

temporal) and the frequency domain. So, a bound like $\|P_N f\|_{L^2} \lesssim N^{-\varepsilon} Y$ can appear, since $\widehat{P_N f}$ being localised in larger dyadic annuli as $N \to \infty$ results in $P_N f$ being localised in smaller and smaller sets. The following lemma is a useful property of frequency localisation, which can be viewed as a manifestation of the uncertainty principle:

Lemma 2.1.2 (Bernstein's inequality)**.** *Let $1 \leq p \leq q \leq \infty$ and $f \in L^1 \cap L^2$. We have*

$$\|P_{\leq N} f\|_{L^q(\mathbb{T}^d)} \lesssim N^{d(\frac{1}{p}-\frac{1}{q})} \|P_{\leq N} f\|_{L^p(\mathbb{T}^d)}. \tag{2.1.1}$$

Proof. Let $f \in \mathcal{S}(\mathbb{T}^d)$. Since $\widehat{P_{\leq N} f}(n) = \mathbb{1}_{\leq N}(|n|)\, \widehat{f}(n)$, we may write

$$P_{\leq N} f = K_N * f$$

where $K_N := \mathcal{F}^{-1}(\mathbb{1}_{\leq N})$. This gives

$$K_N(x) = N^d K_1(Nx),$$

which means that

$$\|K_N\|_{L^r} = N^{d(1-\frac{1}{r})} \|K_1\|_{L^r} \lesssim N^{d(1-\frac{1}{r})}.$$

Thus, by Young's convolution inequality, we have

$$\begin{aligned} \|P_{\leq N} f\|_{L^q} &= \|K_N * f\|_{L^q} \\ &\leq \|K_N\|_{L^r} \|f\|_{L^q} \end{aligned}$$

for $1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. This means we have

$$\|P_{\leq N} f\|_{L^q} \lesssim N^{d(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^q}.$$

Now, consider $f \in L^1 \cap L^2$. Since $P_{\leq N} P_{\leq N} f = P_{\leq N} f \in \mathcal{S}$, we may replace f by $P_{\leq N} f$ above to get the desired result. □

2.2 The Duhamel formulation

Recall the NLS

$$i\partial_t u + \Delta u = \mathcal{N}(u)$$

on the torus $\mathbb{T}^d$. We perform a Fourier transform in x to receive

$$\partial_t \widehat{u}(t,n) = -i|n|^2 \widehat{u}(t,n) - i\widehat{\mathcal{N}(u)}(t,n).$$

For a fixed n, this is an ordinary differential equation in t. By variation of parameters, we get

$$\widehat{u}(t,n) = e^{-i|n|^2}\widehat{u}(0,n) - i\int_0^t e^{-i|n|^2(t-t')}\widehat{\mathcal{N}(u)}(t',n)\,dt'.$$

Then, an inverse Fourier transform gives

$$u(t,x) = S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(u)(t',x)\,dt', \tag{2.2.1}$$

meaning that we have transformed the NLS into an integral equation. We call (2.2.1) the *Duhamel formulation* for the NLS. Here, we define the *linear Schrödinger propagator* $S(t)$ to be the Fourier multiplier operator given by

$$\widehat{S(t)f}(n) = e^{-it|n|^2}\widehat{f}(n).$$

For the NLS on $\mathbb{R}^d$, we simply replace n by ξ for this definition, and otherwise the Duhamel formulation is the same. For a given initial data u_0, the solution to the (homogeneous) linear Schrödinger equation is given by $S(t)u_0$, and the solution to the nonlinear Schrödinger equation is given by (2.2.1). For this reason, we will sometimes refer to $S(t)u_0$ as the *linear solution*, and to u satisfying (2.2.1) as the *nonlinear solution*.

The linear Schrödinger propagator enjoys some properties that are direct consequences of the way it is defined. Namely, it is a unitary operator from H^s_x to itself, and the linear solution $S(t)u_0$ belongs in $C_T H^s_x$. That is, for any $u_0 \in H^s_x$ where $s \in \mathbb{R}$, we have

$$\|S(t)u_0\|_{H^s_x} = \|u_0\|_{H^s_x},$$

and so clearly $\|S(t)u_0\|_{C_T H^s_x} = \sup_{t\in[0,T]}\|S(t)u_0\|_{H^s_x} = \|u_0\|_{H^s_x} < \infty$. We will analyse some other properties of $S(t)$, in particular bounds of the form $\|S(t)u_0\|_{L^q_x} = \|u_0\|_{L^r_x}$, in Chapter 4.

Viewing the linear solution as a sum of different frequencies by the inversion formula $S(t)u_0 = \sum_{n\in\mathbb{Z}^d}\widehat{u_0}(n)\,e^{ix\cdot n - it|n|^2}$, we see that each frequency experiences a phase shift dependent on t, which is more extreme for larger frequencies, that is for large $|n|$. In other words, the frequencies of u_0 are dispersed over time. In local time, this dispersion tends to cause destructive interference and cancellation (hence good smoothness or decay) on $\mathbb{R}^d$, since the frequency domain is continuous. On the torus $\mathbb{T}^d$, however, it is not as helpful, since constructive interference can be more problematic on the discrete frequency domain. Figure 2.1 is a visual analogy of this situation. This is the main difficulty in proving well-posedness results on $\mathbb{T}^d$. Bourgain's introduction of the Fourier restriction norm method in [3] is hence highly influential, not only for the results in the paper, but for the new methodologies that came from it. We will discuss this methodology in Chapter 5 and remark on some results that followed in Chapter 8.

Remark 2.2.1. *Although we have $S(0)u_0 = u_0$ almost everywhere (with respect*

to the Lebesgue measure), the initial data u_0 might be very rough and somewhat problematic, and so it is not obvious that $S(t)u_0 \to u_0$ pointwise if it is only assumed that $u_0 \in H^s_x(\mathbb{R}^d)$. See [27] and [35].

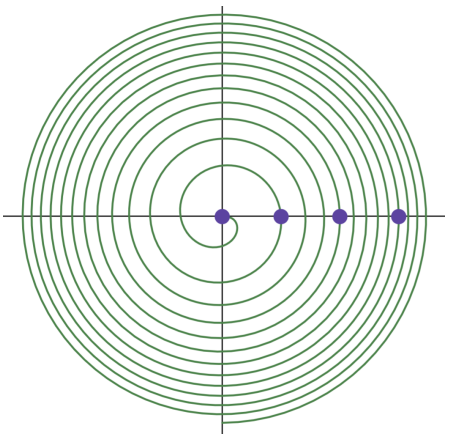


Figure 2.1: The phase shift on frequency components induced by $S(t)$. Here, the polar coordinate plot $\theta = -2\pi r^2$ shows the phase shift $-t|\xi|^2$ as a function of $|\xi|$ (at time $t = 2\pi$). Heuristically, when integrating over frequencies, this twisting is good, since it leads to cancellation. However, when the frequency domain is discrete, so say $n \in \mathbb{Z}$, then this twisting could cause different frequency components (dots above) to align in phase.

2.3 Perturbation

When dealing with the problem of well-posedness of the nonlinear Schrödinger equation, we wish to be able to relate the nonlinear solution to the linear solution with the same initial data. That is, we would like to use the properties of the linear Schrödinger propagator to obtain favourable bounds for a contraction mapping argument, the details of which we will discuss soon.

Heuristically, we expect the nonlinear Schrödinger equation to be well-posed, if nonlinear solutions are "close enough" to some linear solution. In other words, we wish to treat the nonlinear solution as a perturbation of the linear solution. The main concern, then, is to rigorously define this notion of closeness. We do this by norms. In particular, we wish to have the solution given by the Duhamel formulation (2.2.1) to be close in norm to the linear solution, so

$$\|S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(u)\,dt'\| \sim \|S(t)u_0\|.$$

We will see that on $\mathbb{R}^d$, the H^s norm is usually a good way to tackle this problem (see Chapters 3 and 4), although we will have to be more precise when on $\mathbb{T}^d$ (see Chapter 5).

2.4 Contraction mapping arguments

Although we have derived the Duhamel formulation, it tells us little about whether such a u satisfying (2.2.1) exists. For well-posedness, we wish to show that a solution exists and is unique. For this, we rely on the following theorem:

Theorem 2.4.1 (Banach's fixed point theorem)**.** *Let $(X, \|\cdot\|)$ be a Banach space and let $\Gamma : X \to X$ satisfy*

$$\|\Gamma x - \Gamma y\| \leq c\,\|x - y\| \quad \text{for all } x, y \in X,$$

for some $0 < c < 1$. Then Γ has a unique fixed point in X.

This is a standard theorem and can be found in, for instance [30, Chapter 6]. In particular, given $u_0 \in C_T H^s_x$, we will define the solution map

$$\Gamma_{u_0}(u) := S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(u)\,dt'$$

on some ball $B_R \subset C_T H^s_x$, and show that it is indeed a contraction mapping on the closure $\overline{B_R}$. Clearly, the fixed points of the solution map satisfy the Duhamel formulation (2.2.1). In particular, we have to show that $\Gamma_{u_0} : \overline{B_R} \to \overline{B_R}$ and that Γ_{u_0} is Lipschitz continuous in $\overline{B_R}$.

To achieve a full well-posedness result, we are also required to have continuous dependence on initial data. We may achieve this by showing that the solution map is uniformly (Lipschitz) continuous on H^s_x with respect to initial data, since in that case

$$\|u - v\| = \|\Gamma_{u_0}(u) - \Gamma_{v_0}(v)\| \lesssim \|u_0 - v_0\|,$$

for u, v unique fixed points of $\Gamma_{u_0}, \Gamma_{v_0}$ respectively.

By the triangle inequality, we can write

$$\|\Gamma_{u_0}(u)\| \leq \|S(t)u_0\| + \|\int_0^t S(t-t')\mathcal{N}(u)\,dt'\|.$$

So, the problem is mainly controlling the two norms on the right-hand side. The first of these tends to be relatively easy to handle, as we have already seen that $S(t)$ is a unitary operator from H^s_x to itself (and on other function spaces we will consider, it is well-behaved). The second norm on the right is exactly the perturbation from the linear solution and requires more effort. We run into two main hurdles. First, we must find some way to relate the norm of this integral to the norm of the nonlinearity $\mathcal{N}(u) = \pm|u|^{p-1}u$. Second, we must show that the norm of the nonlinearity is comparable in some way to the norm of u directly. In the next chapter, we will see how we can achieve this in a relatively simpler setting.

Chapter 3

Well-posedness of the nonlinear Schrödinger equation for $s > \frac{d}{2}$

We look at a relatively simpler case of the problem. We will prove the following high regularity local well-posedness result:

Theorem 3.0.1. *The nonlinear Schrödinger equation is locally well-posed in $H^s(M^d)$ for $s > \frac{d}{2}$ and $M = \mathbb{R}$ or $\mathbb{T}$.*

Let $0 < T < 1$ be fixed, and $t \in [0, T]$. Recall that we have the following:

- $\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s}$ for all $t \in \mathbb{R}$,
- $\| \int_0^t S(t - t')F \, dt' \|_{C_T H^s_x} \leq T \|F\|_{C_T H^s_x}$.

This means that in the solution map

$$\Gamma_{u_0}(u) = S(t)u_0 \mp i \int_0^t S(t - t')|u|^{p-1}u(t') \, dt',$$

we have control over the size (in H^s norm) of the homogeneous $S(t)u_0$ term and the nonhomogeneous $\mp i \int_0^t S(t-t')|u|^{p-1}u(t') \, dt'$ term. However, we currently lack the necessary tools to compare the nonlinearity itself to the solution u. On the other hand, the regularity condition $s > \frac{d}{2}$ is quite strong, so the nonlinearity ends up being relatively easy to control. This is due to the following lemma:

Lemma 3.0.2. *For $s > \frac{d}{2}$, the Sobolev space $H^s(M^d)$ is an algebra for $M = \mathbb{R}$ or $\mathbb{T}$. That is, given any $f, g \in H^s$, we have*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}.$$

Proof. We will conduct the proof in the Sobolev space $H^s(\mathbb{T}^d)$. The proof for $H^s(\mathbb{R}^d)$ is nearly identical. Notice that if we have $n = n_1 + n_2$, then by the triangle inequality, we have

$$\langle n \rangle^s \leq (\langle n_1 \rangle + \langle n_2 \rangle)^s \lesssim \langle n_1 \rangle^s + \langle n_2 \rangle^s$$

for $s \geq 0$.

Let $s > \frac{d}{2}$. See that

$$\|fg\|_{H^s} = \|\langle n\rangle^s \widehat{fg}(n)\|_{\ell^2_n} = \|\langle n\rangle^s \widehat{f} * \widehat{g}(n)\|_{\ell^2_n} = \|\langle n\rangle^s \sum_{n=n_1+n_2} \widehat{f}(n_1)\,\widehat{g}(n_2)\|_{\ell^2_n}.$$

Using the previous fact and the triangle inequality again, we can write

$$\begin{aligned}\|fg\|_{H^s} &\lesssim \|\sum_{n=n_1+n_2} \langle n_1\rangle^s \widehat{f}(n_1)\,\widehat{g}(n_2)\|_{\ell^2_n} + \|\sum_{n=n_1+n_2} \langle n_2\rangle^s \widehat{f}(n_1)\,\widehat{g}(n_2)\|_{\ell^2_n} \\ &= \|\langle\cdot\rangle^s \widehat{f} * \widehat{g}\|_{\ell^2_n} + \|\widehat{f} * \langle\cdot\rangle^s \widehat{g}\|_{\ell^2_n}.\end{aligned}$$

Applying Young's inequality, we get

$$\begin{aligned}\|\langle\cdot\rangle^s \widehat{f} * \widehat{g}\|_{\ell^2_n} &\leq \|\langle\cdot\rangle^s \widehat{f}\|_{\ell^2_n}\ \|\widehat{g}\|_{\ell^1_n} \\ &= \|f\|_{H^s}\ \|\langle\cdot\rangle^{-s}\langle\cdot\rangle^s \widehat{g}\|_{\ell^1_n} \\ &\leq \|f\|_{H^s}\ \|\langle\cdot\rangle^{-s}\|_{\ell^2_n}\ \|\langle\cdot\rangle^s \widehat{g}\|_{\ell^2_n} \\ &\lesssim_s \|f\|_{H^s}\,\|g\|_{H^s},\end{aligned}$$

where we have used Cauchy–Schwarz and the fact that $(\sum_n \langle n\rangle^{-2s})^{\frac{1}{2}} \lesssim_s 1$ since $s > \frac{d}{2}$. Similarly, we have $\|\widehat{f} * \langle\cdot\rangle^s \widehat{g}\|_{\ell^2_n} \lesssim_s \|f\|_{H^s}\,\|g\|_{H^s}$, which means that

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\,\|g\|_{H^s}$$

as desired. □

In particular, Lemma 3.0.2 implies that

$$\|\,|u|^{p-1}u\,\|_{H^s_x} \lesssim \|u\|^p_{H^s_x} \tag{3.0.1}$$

for $s > \frac{d}{2}$, meaning that we have control over the nonlinearity directly. We proceed to write out the contraction mapping argument in detail.

Proof (of Theorem 3.0.1). Throughout this proof we will denote by c_j the implicit constant coming from any relevant bound or inequality we apply, or relevant products thereof. Let $0 < T < 1$ and $s > \frac{d}{2}$. Define the solution map

$$\Gamma_{u_0}(u) := S(t)u_0 \mp i\int_0^t S(t-t')|u|^{p-1}u(t')\,dt',$$

where $u_0 \in H^s_x(M^d)$, and define the ball

$$B_R := \{u \in C_T H^s_x(\mathbb{R}_t \times M^d_x) : \|u\|_{C_T H^s_x} < R\},$$

where $R > 0$ is a radius which we will choose explicitly later.

Fix $u_0 \in H^s_x(M^d)$. Let $u \in \overline{B_R}$. See that by the triangle inequality we have

$$\|\Gamma_{u_0}(u)\|_{C_T H^s_x} \leq \|S(t)u_0\|_{C_T H^s_x} + \| \int_0^t S(t-t')|u|^{p-1}u(t')\,dt'\|_{C_T H^s_x}.$$

For $t \in [0,T]$, we apply the fact that $S(t)$ is a unitary operator on H^s and the estimate $\| \int_0^t S(t-t')F\,dt'\|_{C_T H^s_x} \leq T\|F\|_{C_T H^s_x}$ to get

$$\|\Gamma_{u_0}(u)\|_{C_T H^s_x} \leq \|u_0\|_{H^s_x} + T\| \, |u|^{p-1}u \, \|_{C_T H^s_x}.$$

Then, 3.0.2 gives

$$\|\Gamma_{u_0}(u)\|_{C_T H^s_x} \leq \|u_0\|_{H^s_x} + c_1 T \|u\|^p_{C_T H^s_x}. \tag{3.0.2}$$

Let $v \in \overline{B_R}$ as well. See that

$$\begin{aligned}\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s_x} &\leq \| \int_0^t S(t-t')(|u|^{p-1}u - |v|^{p-1}v)(t')\,dt'\|_{C_T H^s_x} \\ &\leq T\| \, |u|^{p-1}u - |v|^{p-1}v \, \|_{C_T H^s_x}.\end{aligned}$$

Noting that

$$a^p - b^p = (a-b)\sum_{j+k=p-1} a^j b^k$$

for any $a, b \in \mathbb{R}$, we may similarly write

$$\begin{aligned}|u|^{p-1}u - |v|^{p-1}v &= (u\overline{u})^{\frac{p-1}{2}}u - (v\overline{v})^{\frac{p-1}{2}}v \\ &= (u-v)\overline{u}\,(u\overline{u})^{\frac{p-3}{2}}u + v(\overline{u}-\overline{v})\,(u\overline{u})^{\frac{p-3}{2}}u + \dots \\ &\qquad + (v\overline{v})^{\frac{p-3}{2}}\,v(\overline{u}-\overline{v})u + (v\overline{v})^{\frac{p-3}{2}}\,v\overline{v}(u-v),\end{aligned}$$

so we have

$$\begin{aligned}\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s_x} &\leq T\| \, (u\overline{u})^{\frac{p-1}{2}}u - (v\overline{v})^{\frac{p-1}{2}}v \, \|_{C_T H^s_x} \\ &\leq c_1 T\|u-v\|_{C_T H^s_x}\left(\sum_{j+k=p-1} \|u\|^j_{C_T H^s_x}\,\|v\|^k_{C_T H^s_x}\right)\end{aligned}$$

by the triangle inequality and Lemma 3.0.2. Notice that

$$\|u\|^j_{C_T H^s_x}\,\|v\|^k_{C_T H^s_x} \lesssim \|u\|^{p-1}_{C_T H^s_x} + \|v\|^{p-1}_{C_T H^s_x}$$

for all j, k with $j + k = p - 1$. From $u, v \in \overline{B_R}$ we have $\|u\|_{C_T H^s_x}$, $\|v\|_{C_T H^s_x} \leq R$, which means

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s_x} \leq c_2 T R^{p-1}\|u-v\|_{C_T H^s_x}. \tag{3.0.3}$$

Set $R = 2\|u_0\|_{H^s_x}$, and choose $0 < T < 1$ such that

$$T \leq \frac{1}{2c_1 R^{p-1}} \quad \text{and} \quad T \leq \frac{1}{2c_2 R^{p-1}}.$$

Then, we have from (3.0.2) that

$$\|\Gamma_{u_0}(u)\|_{C_T H^s_x} \leq \frac{R}{2} + \frac{R}{2} = R,$$

and from (3.0.3) that

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{C_T H^s_x} \leq \frac{1}{2}\|u - v\|_{C_T H^s_x}.$$

That is, for a fixed $u_0 \in H^s_x(M^d)$, the solution map Γ_{u_0} is a Lipschitz continuous mapping from $\overline{B_R}$ to itself. Thus, Banach's fixed point theorem implies the existence and uniqueness of the solution to the NLS given initial data $u_0 \in H^s_x(M^d)$.

Finally, let $v_0 \in H^s_x(M^d)$ as well, and denote by u and v the unique fixed points of Γ_{u_0} and Γ_{v_0}, respectively. See that $\|u - v\|_{C_T H^s_x} = \|\Gamma_{u_0}(u) - \Gamma_{v_0}(v)\|_{C_T H^s_x}$, and so

$$\begin{aligned}\|u - v\|_{C_T H^s_x} &\leq \|S(t)(u_0 - v_0)\|_{C_T H^s_x} + \| \int_0^t S(t - t')(|u|^2 u - |v|^2 v)(t')\, dt' \|_{C_T H^s_x} \\ &\leq \|u_0 - v_0\|_{H^s_x} + \frac{1}{2}\|u - v\|_{C_T H^s_x}\end{aligned}$$

by a computation similar to earlier. So, we have

$$\|u - v\|_{C_T H^s_x} \leq 2\|u_0 - v_0\|_{H^s_x},$$

meaning that the solution map is uniformly continuous with respect to initial data. □

Chapter 4

Well-posedness of the nonlinear Schrödinger equation on $\mathbb{R}^d$

In the previous chapter, we proved the local well-posedness of the NLS in $H^s(M^d)$ for $s > \frac{d}{2}$ and $M = \mathbb{T}$ or $M = \mathbb{R}$ (Theorem 3.0.1). This was possible without much trouble, since the regularity condition $s > \frac{d}{2}$ means that any solution in the relevant space is rather well-behaved, especially in higher dimensional settings. We would like to have less restriction on the regularity, and hence we investigate what happens when $s \leq \frac{d}{2}$ when working in $\mathbb{R}^d$. This chapter will largely focus on local well-posedness, though we will discuss a result on global well-posedness.

4.1 Scaling heuristics

We hope to gain some information about the low regularity setting through the symmetries of the NLS. In particular, we look at scaling. The equation

$$i\partial_t u + \Delta u = \mathcal{N}(u)$$

has no direct dependence on x as a term or coefficient. That is, any solution u to the NLS is not controlled by its spatial scale. So, we expect that a solution with its initial value localised in x over a ball of small measure should, when scaled appropriately, behave similarly to a solution with its initial value localised on a ball of a relatively larger measure.

Given a solution u of the NLS in $\mathbb{R}^d$ with $u\big|_{t=0} = u_0$, define the scaled function

$$u^\lambda(t,x) := \frac{1}{\lambda^a}\, u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

with

$$u_0^\lambda(x) := u^\lambda(0,x) = \frac{1}{\lambda^a}\, u_0\left(\frac{x}{\lambda}\right),$$

where λ, a are real with $\lambda > 0$. Here, we are scaling t by $\frac{1}{\lambda^2}$ so as to have the

same scaling after taking one partial derivative in t as opposed to the two partial derivatives in x (the Laplacian). We would like to find $a \in \mathbb{R}$ such that the scaled function u^λ is a solution to the NLS with initial condition u_0^λ for all $\lambda \in \mathbb{R}$. Let $t' := \frac{t}{\lambda^2}$ and $x' := \frac{x}{\lambda}$. Recall that $\mathcal{N}(u) = \pm|u|^{p-1}u$ for some odd $p \in \mathbb{N}$, and so u satisfies

$$i\partial_t u + \Delta u = \pm|u|^{p-1}u,$$

which means that

$$i\partial_t u^\lambda + \Delta u^\lambda = \frac{i}{\lambda^{a+2}}\,\partial_t u(t', x') + \frac{1}{\lambda^{a+2}}\,\Delta u(t', x') = \pm\frac{1}{\lambda^{a+2}}\,|u|^{p-1}u(t', x'),$$

which we would hope to have be equal to

$$\pm|u^\lambda|^{p-1}u^\lambda = \pm\frac{1}{\lambda^{ap}}\,|u|^{p-1}u(t', x'),$$

and this is the case if and only if $a + 2 = ap$, that is $a = \frac{2}{p-1}$, which can only happen when $p > 1$. Notice that we pick up a scaling of $\frac{1}{\lambda^{a+2}}$ from the (linear) derivative operators, whereas the scaling of $\frac{1}{\lambda^{ap}}$ comes from the nonlinearity in the equation. Heuristically, we see that the scaled function u^λ is a solution to the NLS if and only if the scaling preserves some balance between linear dispersion and nonlinear concentration.

Let $s \in \mathbb{R}$, and for some $f \in \dot{H}^s(\mathbb{R}^d)$, let

$$f^\lambda(x) := \frac{1}{\lambda^{\frac{2}{p-1}}}\,f\left(\frac{x}{\lambda}\right).$$

Noting that

$$\widehat{f^\lambda}(\xi) = \lambda^{d-\frac{2}{p-1}}\,\widehat{f}(\lambda\xi),$$

we have

$$\begin{aligned}
\|f^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |\xi|^{2s}\,|\widehat{f^\lambda}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \\
&= \lambda^{d-\frac{2}{p-1}-\frac{d}{2}-s}\left(\int_{\mathbb{R}^d} |\lambda\xi|^{2s}\,|\widehat{f}(\lambda\xi)|^2\,d(\lambda^d\xi)\right)^{\frac{1}{2}} \\
&= \lambda^{\frac{d}{2}-\frac{2}{p-1}-s}\,\|f\|_{\dot{H}^s(\mathbb{R}^d)}.
\end{aligned}$$

So, let $s_c := \frac{d}{2} - \frac{2}{p-1}$ denote the scaling-critical Sobolev index. Note that $s_c < \frac{d}{2}$. See that the above means

$$\|u_0^\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s_c-s}\,\|u_0\|_{\dot{H}^s(\mathbb{R}^d)}$$

for any $u_0 \in \dot{H}^s(\mathbb{R}^d)$. We may heuristically consider u_0 to evolve over time, getting scaled in a manner similar to increasing λ as t grows. If $\widehat{u_0}$ decays rapidly as

$|\xi|$ increases (so if s is large), then the dispersion is strong enough that any additional high frequencies introduced by concentration will largely face destructive interference and cancellation. On the other hand, if u_0 has low regularity, then the high frequencies coming from the concentration can overcome the cancellation effects brought on by the dispersion, so $\|u\|_{\dot{H}^s_x(\mathbb{R}^d)}$ increases over time, potentially becoming infinite. The above shows that the turning point is at $s = s_c$.

We have three types of regularity with respect to scaling:

- Subcritical regularity ($s > s_c$): If u is a solution to the NLS on $t \in [0, T]$, then u^λ is a solution to the NLS on $t \in [0, \lambda^2 T]$. The initial data u_0^λ is smaller than u_0 in $\dot{H}^s$ norm. That is, smaller initial data allows for solutions to live longer. Since by scaling, we can reduce to the small data setting, we expect to have local well-posedness.

- Critical regularity ($s = s_c$): Scaling has no effect on the $\dot{H}^s$ norm of the initial data, meaning that further analysis is needed.

- Supercritical regularity ($s < s_c$): According to the same heuristic, larger initial data allows for solutions to live longer. In some sense, this is too good to be true, especially since we expect solutions to suffer from concentration and possibly blow up in the low regularity setting. Thus, we expect to have ill-posedness.

The above outlines our general expectations of local well-posedness when we consider $s \leq \frac{d}{2}$. In particular, we should expect the cubic NLS ($p = 3$) to be locally well-posed in H^s for $s > -\frac{1}{2}$ when $d = 1$, and for $s > 0$ when $d = 2$. It is also worth noting that the scaling-critical Sobolev index $s_c = \frac{d}{2} - \frac{2}{p-1}$ is only a small improvement from $\frac{d}{2}$ when the nonlinearity is more extreme (large p).

4.2 Strichartz estimates

The main problem is to obtain a nonlinear (or multilinear) estimate like (3.0.1), so that we can carry out the contraction mapping argument in the same way as we did when $s > \frac{d}{2}$ in Chapter 3. We will achieve this by Strichartz estimates. We outline the general approach here since our main focus is the NLS on $\mathbb{T}^d$, but an in-depth discussion can be found in [32, Chapter 2.3]. Along the way, we will discuss heuristics to set our expectations for what is to come in Chapter 5 and onwards.

Notice that by definition,

$$\widehat{S(t)u_0}(\xi) = e^{-it|\xi|^2}\, \widehat{u_0}(\xi),$$

which means that we have

$$S(t)u_0(x) = (K_t * u_0)(x),$$

where

$$K_t(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-it|\xi|^2 + ix\cdot\xi}\, d\xi \tag{4.2.1}$$

is the inverse Fourier transform of the Fourier multiplier $e^{-it|\xi|^2}$. We can then write

$$\begin{aligned} -t|\xi|^2 + x\cdot\xi &= -t\left(|\xi|^2 - 2\frac{x}{2t}\cdot\xi + \frac{|x|^2}{4t^2} - \frac{|x|^2}{4t^2} \right) \\ &= -t\left|\xi - \frac{x}{2t}\right|^2 + \frac{|x|^2}{4t} \end{aligned}$$

to get

$$\begin{aligned} K_t(x) &= \frac{1}{(2\pi)^d}\, e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{-it|\xi - \frac{x}{2t}|^2}\, d\xi \\ &= \frac{1}{(2\pi)^d}\, e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{-it|\zeta|^2}\, d\zeta \\ &= \frac{1}{(4\pi i t)^{\frac{d}{2}}}\, e^{-\frac{|x|^2}{4it}} \end{aligned}$$

with a change of variables $\zeta := \xi - \frac{x}{2t}$, where we are using the standard branch of the complex square root to define $(4\pi i t)^{\frac{1}{2}}$. That is to say that

$$\begin{aligned} S(t)u_0(x) &= (K_t * u_0)(x) \\ &= \frac{1}{(4\pi i t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4it}}\, u_0(y)\, dy. \end{aligned} \tag{4.2.2}$$

Note that $K_t \in \mathcal{S}_x$. Here, we see that if u_0 has compact support (and is thus absolutely integrable) but is not smooth in x, the linear solution $S(t)u_0$ for any $t \neq 0$ is smooth in x regardless. On the other hand, $S(t)u_0$ will cease to have compact support. This phenomenon is named *instantaneous smoothing*.

This makes sense heuristically, since we expect the propagator $S(t)$ to disperse higher frequencies in u_0 significantly more than lower frequencies. Features such as discontinuities that prevent smoothness happen under insufficient decay of high frequencies. We may imagine for instance, when $t = 0$, that at a point x_0, frequency components of u_0 experience a large amount of constructive interference, causing a discontinuity. But because these high frequencies are dispersed, $S(t)u_0$ when $t \neq 0$ does not suffer from as much constructive interference, so that it becomes smooth at x_0. Conversely, a solution can lose regularity while gaining localisation for a point in time. We can see this from time reversibility (that is, if $u(t,x)$ is a solution, then so is $\overline{u(-t,x)}$), or by considering the solution $v = S(-1)u$ with initial data $v\big|_{t=0} = v_0 := S(-1)u_0$ as $t \to 1^-$. Meanwhile, on $\mathbb{T}^d$, instantaneous smoothing fails (see Figure 4.1).

Recalling the heuristics discussed in Chapter 2.3 and the scaling heuristics

in Chapter 4.1, we should expect solutions of the NLS to be well-behaved and preserve regularity over some time if they are close enough to $S(t)u_0$ in the norm sense for an appropriate norm, since then instantaneous smoothing effects will not be overpowered by nonlinear concentration effects (at least on some local time interval $t \in [0, T]$). Of course, this is all somewhat vague. Nevertheless, it shows that there is some hope that local well-posedness holds in low regularity settings.

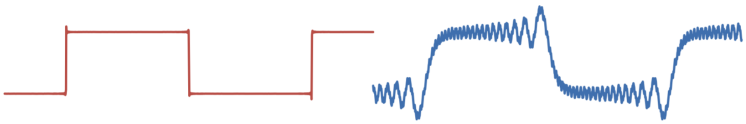


Figure 4.1: Dispersion on $\mathbb{T}$. On the left is a square wave, say u_0, and on the right is the dispersed square wave, so $S(t)u_0$ for some small t (about one and a half period is shown for both functions). Through dispersion, u_0 gains continuity, but potentially loses almost-everywhere differentiability.

Remark 4.2.1. *It is worth noting that the integral in (4.2.1) is not absolutely convergent. We may, however, resolve this by instead writing*

$$K_t(x) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d} e^{-it|\xi|^2 + ix\cdot\xi}\, e^{-\varepsilon|\xi|^2}\, d\xi,$$

where the integral inside the limit is always absolutely convergent. The limit itself exists in the sense of tempered distributions. For a general discussion on tempered distributions, see [10, Chapter 2.3].

Another point to highlight here is that, although we are developing these heuristics, and we have results such as (4.2.2), it is not obvious that $S(t)u_0 \to u_0$ pointwise as $t \to 0$, as we mentioned earlier in Remark 2.2.1.

Most importantly for local well-posedness on $\mathbb{R}^d$ though, from (4.2.2), we directly receive the dispersive estimate

$$\|S(t)u_0\|_{L^\infty_x} \lesssim \frac{1}{|t|^{\frac{d}{2}}} \|u_0\|_{L^1_x}, \quad t \neq 0, \tag{4.2.3}$$

by Young's convolution inequality. Recalling that $S(t)$ is an L^2-unitary operator ($H^0 = L^2$), we have

$$\|S(t)u_0\|_{L^2_x} = \|u_0\|_{L^2_x}. \tag{4.2.4}$$

We may then apply the Riesz–Thorin interpolation lemma (Lemma 2.0.2) to (4.2.3) and (4.2.4) to get

$$\|S(t)u_0\|_{L^p_x} \lesssim \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{p})}} \|u_0\|_{L^{p'}_x}, \quad t \neq 0, \tag{4.2.5}$$

for $2 \leq p \leq \infty$, where p' is the Hölder conjugate of p satisfying $\frac{1}{p'} + \frac{1}{p} = 1$. The estimate (4.2.5) suggests that the operator $S(t)$ preserves the L^p_x norm only for $p = 2$. This is indeed the case, though further justification is needed, see [32, Chapter 2.3]. From the above estimates, we can derive the following, crucial, Strichartz estimates:

Proposition 4.2.2 (Strichartz estimates on $\mathbb{R}^d$ [8, 17, 29, 36])**.** *Let $d \geq 1$. We call a pair of exponents (q, r)* admissible *or* Schrödinger admissible *if we have $2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 2)$. Then, for admissible pairs of exponents (q, r) and $(\tilde{q}, \tilde{r})$ we have:*

- *for $f \in L^2_x(\mathbb{R}^d)$, the homogeneous*

$$\|S(t)f\|_{L^q_t L^r_x(\mathbb{R}_t \times \mathbb{R}^d_x)} \lesssim_{d,q,r} \|f\|_{L^2_x(\mathbb{R}^d)},$$

- *and for $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}_t \times \mathbb{R}^d_x)$, the dual homogeneous*

$$\| \int_{\mathbb{R}} S(-t')F(t')\, dt' \|_{L^2_x(\mathbb{R}^d)} \lesssim_{d,\tilde{q},\tilde{r}} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}_t \times \mathbb{R}^d_x)}$$

- *and the nonhomogeneous*

$$\| \int_0^t S(t-t')F(t')\, dt' \|_{L^q_t L^r_x(\mathbb{R}_t \times \mathbb{R}^d_x)} \lesssim_{d,q,r,\tilde{q},\tilde{r}} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}_t \times \mathbb{R}^d_x)}$$

Strichartz estimates, where for any exponent q, the exponent q' is its Hölder conjugate satisfying $\frac{1}{q'} + \frac{1}{q} = 1$.

The main takeaway here is that locally in time, dispersion by $S(t)$ allows some gain in integrability for f when averaged in time in the L^q_t sense. So, given initial data in L^2_x, the solution u will be in $L^q_t L^r_x$ for some $r > 2$ for most of the time. With consideration of instantaneous smoothing (which is in trade with compact support), the Strichartz estimates also tell us that the solution u will have its L^r_x norm decay as $t \to \infty$, again in the L^q_t-averaged sense.

We will give a partial proof of Proposition 4.2.2 later. Temporarily assuming this result, we approach the cubic NLS on $\mathbb{R}$. Recall that $C_T = L^\infty_t([0, T])$. Choosing $(q, r) = (\infty, 2)$ or $(8, 4)$, and $(\tilde{q}, \tilde{r}) = (8, 4)$, see that we have

$$\| \int_0^t S(t-t')|u|^2 u\, dt' \|_{L^q_t L^r_x} \lesssim \| |u|^2 u \|_{L^{\frac{8}{7}}_t L^{\frac{4}{3}}_x} \tag{4.2.6}$$

by the nonhomogeneous Strichartz estimate in Proposition 4.2.2. Here, and throughout this discussion, we use L^q_t to mean the restriction $L^q_t([0, T])$. Since

$$\frac{7}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \quad \text{and} \quad \frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4},$$

we may apply Hölder's inequality several times to (4.2.6) and receive

$$\begin{aligned} \| \int_0^t S(t-t')|u|^2 u \, dt' \|_{L_t^q L_x^r} &\lesssim \| \mathbb{1}_{[0,T]} \, |u|^2 u \|_{L_t^{\frac{8}{7}} L_x^{\frac{4}{3}}} \\ &\leq T^{\frac{1}{2}} \| |u|^2 u \|_{L_t^{\frac{8}{3}} L_x^{\frac{4}{3}}} \\ &\leq T^{\frac{1}{2}} \|u\|_{L_t^8 L_x^4}^3 \end{aligned} \tag{4.2.7}$$

on $t \in [0,T]$. This means that if we define the space $X_T(\mathbb{R}_t \times \mathbb{R}_x^d)$ with the norm

$$\|f\|_{X_T} := \|f\|_{C_T H_x^s} + \|f\|_{L_t^8 L_x^4},$$

that is $X_T = C_T H_x^s \cap L_t^8 L_x^4$, we can run a contraction mapping argument on the closure of the ball

$$B_R := \{u \in X_T : \|u\|_{X_T} < R\}$$

with an appropriate choice of radius R similar to before. This will thus give the following theorem:

Theorem 4.2.3. *The cubic nonlinear Schrödinger equation is locally well-posed in* $L^2(\mathbb{R})$.

The proof of Theorem 4.2.3 is similar to the proof of Theorem 3.0.1, and thus we will skip the details. The main qualitative difference of note is that in defining the ball for the contraction mapping argument, we require the solution u to have some extra integrability (by being in $L_t^8 L_x^4$).

One may similarly derive other local well-posedness results, potentially converting the estimates in Proposition 4.2.2 to those concerning Sobolev spaces whenever needed. We look at a pair of interesting results. Consider the cubic NLS on $\mathbb{R}^2$. We pick $(q,r) = (\infty, 2)$ or $(4,4)$, and $(\tilde{q}, \tilde{r}) = (4,4)$. A similar computation to (4.2.7) gives

$$\| \int_0^t S(t-t')|u|^2 u \, dt' \|_{L_t^q L_x^r} \lesssim \| |u|^2 u \|_{L_{t,x}^{\frac{4}{3}}} \leq \|u\|_{L_{t,x}^4}^3$$

on $t \in [0,T]$. Now, note that here, we actually have no dependence on T, in contrast to (4.2.7). Recall that in the proof of Theorem 3.0.1 (or 4.2.3), we required some factor of T^θ with $\theta > 0$ to ensure that the solution map Γ_{u_0} is a contraction. In this case, we may ensure the same, if we let $0 < \eta \ll 1$ and stipulate

$$\|u_0\|_{L_x^2} = \eta, \tag{4.2.8}$$

or observe that

$$\lim_{T \to 0^+} \| \mathbb{1}_{[0,T]} \, S(t) u_0 \|_{L_{t,x}^4} = 0,$$

by the Dominated Convergence Theorem, and thus stipulate that

$$T \text{ is small enough such that } \|S(t)u_0\|_{L^4_{t,x}} \leq \eta \tag{4.2.9}$$

to conduct a contraction mapping argument with $R = 2\eta$, where we choose η appropriately. Notice that the stipulation (4.2.8) has no dependence on T (that is, we may consider $T \to \infty$), meaning that we have the following small-data general well-posedness result, whose proof we similarly omit:

Theorem 4.2.4. *There exists some $\eta \ll 1$ such that the cubic nonlinear Schrödinger equation is globally well-posed in $L^2(\mathbb{R}^2)$ if $\|u_0\|_{L^2} \leq \eta$.*

Recall the scaling heuristics from earlier. We discussed that in the case of subcritical regularity ($s > s_c = \frac{d}{2} - \frac{2}{p-1}$), we expect small initial data to allow solutions to live longer. We see that here, we may even choose initial data small enough for solutions to live indefinitely (even though we are in the critical space $L^2_x = H^0_x$!).

On the other hand, from the stipulation (4.2.9) we receive:

Theorem 4.2.5. *The cubic nonlinear Schrödinger equation is locally well-posed in $L^2(\mathbb{R}^2)$.*

Here, it is important to note that the local well-posedness depends on the profile of the initial data u_0, since different initial data will naturally have different L^4_x norms. So, although we have some local well-posedness in the critical space L^2_x, it differs from other local well-posedness results we consider in terms of this dependence. The proof of Theorem 4.2.5 we omit, though there are some differences to the proof of Theorem 3.0.1. In particular, one runs the contraction mapping argument in $L^4_{t,x}$, and then shows that the unique solution $u = \Gamma_{u_0}(u)$ must belong in $C_T L^2_x$. Theorems 4.2.4 and 4.2.5 are somewhat weak results, and much stronger results are available in the defocusing regime, see [7] and Chapter 8.

Now, we return to partially prove the Strichartz estimates in Proposition 4.2.2. We will make use of the following lemma, which directly gives the nonhomogeneous estimate from a similar, non-localised estimate:

Lemma 4.2.6 (Christ–Kiselev lemma for the Schrödinger propagator [5])**.** *Suppose that $0 < T < 1$, and $1 \leq q < \tilde{q}' \leq \infty$ are such that*

$$\| \int_{\mathbb{R}} S(t-t')F(t')\,dt' \|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

for some $1 \leq r, \tilde{r}' \leq \infty$ and $t \in [0,T]$. Then, the estimate

$$\| \int_0^t S(t-t')F(t')\,dt' \|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

holds.

The above is a corollary of a more abstract lemma about localising operators in time, from [5] (see also [28], [33]), that holds for any kernel $K \in C^0(I \times I \to B(X \to Y))$ of bounded operators between Banach spaces X and Y, where I is some time interval. It, along with the dispersive estimate (4.2.5), allows us to prove Proposition 4.2.2 in the non-endpoint case ($q, \tilde{q} \neq 2$).

Partial proof (of Proposition 4.2.2). We will prove the Strichartz estimates in the case where $q, \tilde{q} \neq 2$. We will employ the TT^* method. That is, for a linear operator T, the following statements are equivalent (see [34]):

- T is bounded,
- its adjoint operator T^* is bounded,
- their composition TT^* is bounded.

In this case we will use $T := S(t) : L^2_x \to L^q_t L^r_x$, which means we have that $T^* : L^{q'}_t L^{r'}_x \to L^2_x$ is the operator defined by

$$T^* F = \int_{\mathbb{R}} S(-t')F(t')\,dt',$$

and that it suffices to show their composition

$$TT^* F(t) = \int_{\mathbb{R}} S(t-t')F(t')\,dt'$$

is bounded $L^{q'}_t L^{r'}_x \to L^q_t L^r_x$.

Let $(q, r), (\tilde{q}, \tilde{r})$ be admissible. We have from Minkowski's integral inequality, (4.2.5), and the Hardy–Littlewood–Sobolev inequality (2.0.2) that

$$\begin{aligned}
\| \int_{\mathbb{R}} S(t-t')F(t')\,dt' \|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} &\leq \| \int_{\mathbb{R}} \|S(t-t')F(t')\|_{L^r_x(\mathbb{R}^d)}\,dt' \|_{L^q_t(\mathbb{R})} \\
&\lesssim \| \int_{\mathbb{R}} \frac{1}{|t-t'|^{d(\frac{1}{2}-\frac{1}{r})}} \|F(t')\|_{L^{r'}_x}\,dt' \|_{L^q_t} \\
&= \| \frac{1}{|\cdot|^{d(\frac{1}{2}-\frac{1}{r})}} * \|F(\cdot)\|_{L^{r'}_x} \|_{L^q_t} \\
&\lesssim \|F\|_{L^{q'}_t L^{r'}_x}
\end{aligned}$$

for $2 < q, r$ and for any $F \in L^{q'}_t L^{r'}_x(\mathbb{R} \times \mathbb{R}^d)$, where by q', r' we denote the Hölder conjugates of q, r. This means that the operator TT^* is bounded $L^{q'}_t L^{r'}_x \to L^q_t L^r_x$, and the homogeneous and dual homogeneous Strichartz estimates follow. Composing the two homogeneous estimates, we get

$$\| \int_{\mathbb{R}} S(t-t')F(t')\,dt' \|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

and so the nonhomogeneous estimate follows from the Christ–Kiselev lemma 4.2.6.

For the proof of the endpoint cases, see [17]. □

Chapter 5

Fourier restriction norm spaces

When tackling the problem of local well-posedness of NLS on the torus $\mathbb{T}^d$ for $s \leq \frac{d}{2}$, we cannot use the same reasoning to conclude the contraction mapping argument. This is because Lemma 3.0.2 does not hold, and so we cannot directly bound the nonlinear term. We resolved this problem for the cubic NLS on $\mathbb{R}^d$ for $d = 1, 2$ by using Strichartz estimates as discussed in Chapter 4.2. However, these estimates heavily relied on the dispersive estimate (4.2.3), which does not hold in the periodic setting. As such, we need an alternative method to establish Strichartz estimates, which will allow us to deal with the nonlinearity.

Why does the dispersive estimate fail on the torus $\mathbb{T}^d$? It is easy to see that, for instance on $\mathbb{T}$, the linear solution

$$S(t)u_0 = \sum_{n\in\mathbb{Z}} e^{ixn-it|n|^2}\, \widehat{u_0}(n) \tag{5.0.1}$$

is periodic in t for any fixed x, for any suitable initial data u_0. So, the decay in L^∞_x and conservation in L^2_x norms do not share the same relations as they do when we look at the NLS on $\mathbb{R}$. This is a consequence of $\mathbb{T}$ being compact, since the Fourier transform $\mathbb{T} \to \mathbb{Z}$ is an operator that lands in a discrete and countable space in this case.

One (rough) heuristic for this is to consider the NLS as describing how the surface of water evolves over time. We may think, then, that the initial data u_0 describes some initial distribution of water. If we have a finite amount of water on $\mathbb{R}^d$, an infinite space (not compact), then over time, the highest water level will decrease as the water spreads thin. In contrast, on some spherical planet of finite size (compact; one may alternatively imagine a toroidal planet), the highest water level can never go below some average water level, and the water waves will keep moving back and forth. Figure 5.1 illustrates this heuristic. So, we should not expect to have dispersive estimates on $\mathbb{T}^d$, at least not in the same way.

Recalling the discussion in Chapter 2.3, we reconsider our function spaces. We wish to define spaces better-suited to the NLS than the Sobolev spaces H^s we have been taking advantage of so far. Our critical observation comes from applying a Fourier transform in the time variable in addition to the space variable. Consider

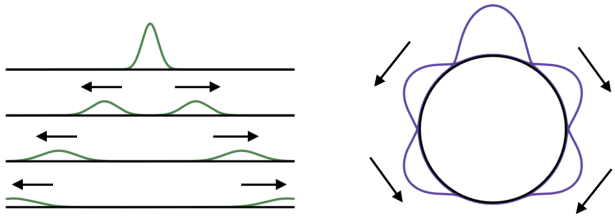


Figure 5.1: The "water on a planet" heuristic. On the left is the evolution of a possible solution on $\mathbb{R}$, and on the right is the evolution of a similar solution in $\mathbb{T}$.

the solution u to the linear Schrödinger equation. It satisfies

$$i\partial_t u + \Delta_x u = 0$$

with $u\big|_{t=0} = u_0$. Taking a space-time Fourier transform on both sides of the linear equation above, we get

$$\begin{aligned}\mathcal{F}_{t,x}(i\partial_t u + \Delta_x u)\,(\tau, n) &= \int_{\mathbb{R}} (i\partial_t \widehat{u}(t,n) - |n|^2 \widehat{u}(t,n))\, e^{-it\tau}\, dt \\ &= -(\tau + |n|^2)\, \widehat{u}(\tau, n) \\ &= 0 = \mathcal{F}_{t,x}(0).\end{aligned}$$

More concisely, we have

$$(\tau + |n|^2)\, \widehat{u}(\tau, n) = 0, \tag{5.0.2}$$

which implies that the space-time Fourier transform of u is a measure supported on the characteristic hypersurface $\{(\tau, n) \in \mathbb{R} \times \mathbb{T}^d : \tau = -|n|^2\}$. In particular, we have

$$\widehat{u}(\tau, n) = \int_{\mathbb{R}} e^{-it|n|^2}\, \widehat{u_0}(n)\, e^{-it\tau} dt = \int_{\mathbb{R}} e^{-it(\tau+|n|^2)} dt\, \widehat{u_0}(n) = \delta(\tau + |n|^2)\, \widehat{u_0}(n), \tag{5.0.3}$$

where δ denotes the Dirac delta function.

We see that the space-time Fourier transform of u is supported on a hypersurface analogous to a paraboloid, see Figure 5.2. Viewing the nonlinear solution as a perturbation of the linear solution, we expect the nonlinear solution to have its space-time Fourier transform supported on a set that is close to this hypersurface, at least for some short time (a solution may not even exist in global time). We wish to take advantage of this localisation. In particular, we expect to be able to bound sums or integrals over some set E, by the measure of E. If, say $E_\tau \subset \{(\tau, n) \in \mathbb{R} \times \mathbb{T}^d : \tau = -|n|^2 + O(1)\}$, then we can count lattice points to estimate $\#E_\tau$. We will see more concrete examples of this kind of bound in Chapter 6.1 and Chapter 7.1.

We define the Fourier restriction norm spaces:

Definition 5.0.1 (Fourier restriction norm spaces)**.** *Let $s, b \in \mathbb{R}$. We denote by $X^{s,b}_{\tau=-|n|^2}(\mathbb{R} \times \mathbb{T}^d)$ the closure of the space of Schwartz functions $\mathcal{S}(\mathbb{R} \times \mathbb{T}^d)$ under*

the norm

$$\|u\|_{X^{s,b}_{\tau=-|n|^2}} := \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \, \widehat{u}(\tau, n)\|_{\ell^2_n L^2_\tau}. \tag{5.0.4}$$

We will abbreviate $X^{s,b}_{\tau=-|n|^2}$ to $X^{s,b}$ whenever the underlying hypersurface is clear from the context.

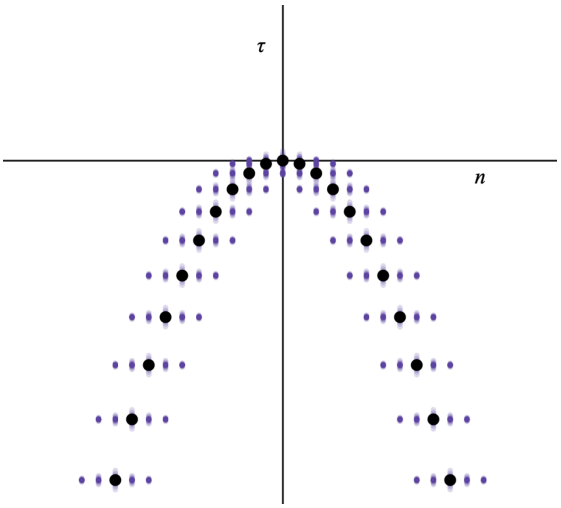


Figure 5.2: Roughly localising around the paraboloid.

Notice that the term $\langle \tau + |n|^2\rangle^b$ in (5.0.4) grows large going away from the characteristic hypersurface of the linear solution for $b > 0$. So, any bound on the $X^{s,b}$ norm that we impose on the nonlinear solution will force its space-time Fourier transform to nearly vanish outside of this hypersurface. In that sense, the $X^{s,b}$ norm penalises any function that lives far away from the hypersurface.

For any $u \in X^{s,b}$, we denote by $U := S(-t)u$ its *interaction representation*. Notice that

$$\widehat{U}(\tau, n) = \int_{\mathbb{R}} e^{it|n|^2} \, \widehat{u}(t, n) \, e^{-it\tau} dt = e^{-i(\tau - |n|^2)} \, \widehat{u}(\tau, n) = \widehat{u}(\tau - |n|^2, n),$$

so we have

$$\begin{aligned} \|u\|_{X^{s,b}} &= \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \, \widehat{u}(\tau, n)\|_{\ell^2_n L^2_\tau} \\ &= \|\langle n\rangle^s \langle \tau'\rangle^b \, \widehat{U}(\tau', n)\|_{\ell^2_n L^2_{\tau'}} \\ &= \|\langle \nabla\rangle^s \langle \partial_t\rangle^b \, U\|_{L^2_{t,x}}, \end{aligned} \tag{5.0.5}$$

where we have used a change of variables with $\tau' = \tau + |n|^2$. We see that for the interaction representation U, the $X^{s,b}$ spaces coincide with the spaces $H^s_x H^b_t$. So, it makes sense to view $X^{s,b}$ spaces as capturing some sort of regularity in the t, x variables for $u \in X^{s,b}$. Specifically, we have that a derivative loss of 2 in x has a similar effect to a derivative loss of 1 in t. This means that we should expect, say $X^{\frac{1}{4},\frac{1}{4}}$, to behave similarly to $X^{0,\frac{3}{8}}$.

The $X^{s,b}$ spaces exhibit the following properties that are direct consequences of standard properties of L^2 spaces and the Fourier transform, which we will use often:

Lemma 5.0.2. *Let $u \in X^{s,b}$. We have:*

1. *$X^{s_1,b_1} \subset X^{s_2,b_2}$ whenever $s_1 \geq s_2$ and $b_1 \geq b_2$,*
2. *$\|u\|_{X^{s,b}_{\tau=-|n|^2}} = \|\overline{u}\|_{X^{s,b}_{\tau=|n|^2}}$,*
3. *$(X^{s,b}_{\tau=-|n|^2})^* = X^{-s,-b}_{\tau=|n|^2}$,*
4. *$X^{s,b}$ is invariant under translation in space and time.*

The following lemma shows that running a contraction mapping argument on $X^{s,b}$ is indeed useful, in particular for $b > \frac{1}{2}$, since it will give us a local well-posedness result in H^s_x:

Lemma 5.0.3. *Let $s, b \in \mathbb{R}$. If $b > \frac{1}{2}$, then we have the embedding*

$$X^{s,b} \subset C_T H^s_x.$$

Proof. Let $u \in X^{s,b}$. See that

$$\|u(t)\|_{H^s_x} = \|\langle n\rangle^s \widehat{u}(t,n)\|_{\ell^2_n} \leq \|\langle n\rangle^s \int_{\mathbb{R}} |\widehat{u}(\tau,n)|\, d\tau\|_{\ell^2_n}$$

since we have $|\widehat{u}(t,n)| = |\int_{\mathbb{R}} \widehat{u}(\tau,n)\, e^{it\tau}\, d\tau| \leq \int_{\mathbb{R}} |\widehat{u}(\tau,n)|\, d\tau$ where we used the Fourier inversion formula in time. We then have

$$\begin{aligned}\|\langle n\rangle^s \int_{\mathbb{R}} |\widehat{u}(\tau,n)|\, d\tau\|_{\ell^2_n} &= \|\langle n\rangle^s \int_{\mathbb{R}} \langle \tau+|n|^2\rangle^b \langle \tau+|n|^2\rangle^{-b} |\widehat{u}(\tau,n)|\, d\tau\|_{\ell^2_n} \\ &\leq \left\| \langle n\rangle^s \|\langle \tau+|n|^2\rangle^{-b}\|_{L^2_\tau} \|\langle \tau+|n|^2\rangle^b \widehat{u}(\tau,n)\|_{L^2_\tau} \right\|_{\ell^2_n}\end{aligned}$$

by Cauchy–Schwarz, and since $\int_{\mathbb{R}} \langle \tau+|n|^2\rangle^{-2b}\, d\tau \lesssim 1$ for $b > \frac{1}{2}$, we have

$$\begin{aligned}\left\| \langle n\rangle^s \|\langle \tau+|n|^2\rangle^{-b}\|_{L^2_\tau} \|\langle \tau+|n|^2\rangle^b \widehat{u}(\tau,n)\|_{L^2_\tau} \right\|_{\ell^2_n} &\lesssim \left\| \langle n\rangle^s \|\langle \tau+|n|^2\rangle^b \widehat{u}(\tau,n)\|_{L^2_\tau} \right\|_{\ell^2_n} \\ &= \|\langle n\rangle^s \langle \tau+|n|^2\rangle^b \widehat{u}(\tau,n)\|_{\ell^2_n L^2_\tau} \\ &= \|u\|_{X^{s,b}}.\end{aligned}$$

Thus, we have $\|u(t)\|_{H^s_x} \lesssim \|u\|_{X^{s,b}}$ uniformly in t. It remains to show that u is continuous in t. Let $h \in \mathbb{R}$. Again, using the Fourier inversion formula in time,

we have

$$\begin{aligned}\|u(t+h)-u(t)\|_{H^s_x} &= \|\langle n\rangle^s \int_{\mathbb{R}} \widehat{u}(\tau,n)\,(e^{i(t+h)\tau}-e^{it\tau})\,d\tau\|_{\ell^2_n}\\ &= \|\langle n\rangle^s \int_{\mathbb{R}} \widehat{u}(\tau,n)\,e^{it\tau}(e^{ih\tau}-1)\,d\tau\|_{\ell^2_n}\\ &\lesssim \|u\|_{X^{s,b}}\end{aligned}$$

by a similar computation to the above, also using the fact that $|e^{ih\tau}-1|\le 2$ for all h. Since $e^{ih\tau}-1\to 0$ as $h\to 0$ and the bound above is uniform, we have from the dominated convergence theorem that u is continuous with respect to the time variable and thus we conclude that $u\in C_T H^s_x$. □

In addition to the above, we have the property that $X^{s,b}$ is closed under multiplication by a smooth function in time.

Lemma 5.0.4. *Let $s,b\in\mathbb{R}$ and ϕ be a smooth function belonging to $\mathcal{S}_t(\mathbb{R})$. We have*

$$\|\phi(t)\,u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}.$$

Proof. See that by the Fourier inversion formula in time,

$$\begin{aligned}\|\phi(t)\,u\|_{X^{s,b}} &= \|\int_{\mathbb{R}} \widehat{\phi}(\tau_1)\,e^{it\tau_1}\,d\tau_1\,u\|_{X^{s,b}}\\ &\le \int_{\mathbb{R}} \|\widehat{\phi}(\tau_1)\,e^{it\tau_1}\,u\|_{X^{s,b}}\,d\tau_1,\end{aligned}$$

where we have used Minkowski's integral inequality. Notice that

$$\|\widehat{\phi}(\tau_1)\,e^{it\tau_1}\,u\|_{X^{s,b}} \le |\widehat{\phi}(\tau_1)|\,\|e^{it\tau_1}\,u\|_{X^{s,b}}$$

and

$$\mathcal{F}_{t,x}(e^{it\tau_1}\,u)(\tau,n) = \widehat{u}(\tau-\tau_1,n).$$

So, we have

$$\begin{aligned}\int_{\mathbb{R}} \|\widehat{\phi}(\tau_1)\,e^{it\tau_1}\,u\|_{X^{s,b}}\,d\tau_1 &\le \int_{\mathbb{R}} |\widehat{\phi}(\tau_1)|\,\|\langle n\rangle^s\langle \tau+|n|^2\rangle^b\,\widehat{u}(\tau-\tau_1,n)\|_{\ell^2_n L^2_\tau}\,d\tau_1\\ &= \int_{\mathbb{R}} |\widehat{\phi}(\tau_1)|\,\|\langle n\rangle^s\langle \tau'+\tau_1+|n|^2\rangle^b\,\widehat{u}(\tau',n)\|_{\ell^2_n L^2_{\tau'}}\,d\tau_1\\ &\lesssim \int_{\mathbb{R}} |\widehat{\phi}(\tau_1)|\,\|\langle n\rangle^s\langle \tau_1\rangle^{|b|}\langle \tau'+|n|^2\rangle^b\,\widehat{u}(\tau',n)\|_{\ell^2_n L^2_{\tau'}}\,d\tau_1\\ &= \|u\|_{X^{s,b}}\int_{\mathbb{R}} \langle\tau_1\rangle^{|b|}\,|\widehat{\phi}(\tau_1)|\,d\tau_1\\ &\lesssim_\phi \|u\|_{X^{s,b}}\end{aligned}$$

since $\phi\in\mathcal{S}_t(\mathbb{R})$, where we have used the change of variables $\tau'=\tau-\tau_1$, and Peetre's inequality $\langle\tau'+\tau_1+|n|^2\rangle^b \lesssim \langle\tau_1\rangle^{|b|}\langle\tau'+|n|^2\rangle^b$. □

5.1 Linear estimates

To conduct a contraction mapping argument in $X^{s,b}$, we would enjoy having estimates of the sort

1. $\|S(t)u_0\|_{X^{s,b}} \lesssim \|u_0\|_{H^s_x}$, (Homogeneous linear est.)
2. $\| \int_0^t S(t-t')F(t')\,dt'\|_{X^{s,b}} \lesssim T\|F\|_{X^{s,b-1}}$, (Nonhomogeneous linear est.)
3. $\|\mathcal{N}(u)\|_{X^{s,b-1}} \lesssim \|u\|^p_{X^{s,b}}$, (Nonlinear est.)

since they would allow us to control both the linear and the nonlinear terms in the solution map

$$\Gamma_{u_0}(u) = S(t)u_0 \pm i\int_0^t S(t-t')\mathcal{N}(u)(t')\,dt'.$$

Note that since we are integrating with respect to time, we expect there to be some loss in regularity in the time variable in (2). The estimates (1) and (2) are not true exactly as above. We will, however, show that they hold under slight modifications. The nonlinear estimate (3) on the other hand, is more delicate, and as such we will direct our attention to it later in Chapters 6.2 and 7.2.

Disappointingly, the linear solution $u = S(t)u_0$ is not in the space $X^{s,b}$, despite living on the characteristic hypersurface. It follows from (5.0.3) that

$$\begin{aligned}\|u\|_{X^{s,b}} &= \|\langle n\rangle^s\langle \tau+|n|^2\rangle^b\,\widehat{u}(\tau,n)\|_{\ell^2_n L^2_\tau}\\ &= \|\langle n\rangle^s\langle \tau+|n|^2\rangle^b\,\delta(\tau+|n|^2)\,\widehat{u_0}(n)\|_{\ell^2_n L^2_\tau}\\ &= \|\langle n\rangle^s\,\delta(\tau+|n|^2)\,\widehat{u_0}(n)\|_{\ell^2_n L^2_\tau},\end{aligned}$$

which does not exist since $\delta \notin L^2$. To mitigate this, we localise the linear solution in time so that it is in $X^{s,b}$. This is natural, since by the uncertainty principle, the function $S(t)u_0$ which is completely unrestricted in t, x will be completely localised in τ, n. Given that the linear solution can be periodic in time (see (5.0.1)), we cannot expect it to be in L^2_t unless it is uniformly zero. By localising in time, we force the linear solution to be Schwartz in exchange for its localisation in the frequency domain, but this is a profitable tradeoff as we will observe, in particular for obtaining local well-posedness results.

Let $\eta \in \mathcal{S}_t$ be a smooth function such that

- $\eta(t) = 1$ whenever $t \in [-1, 1]$,
- $\eta(t) = 0$ whenever $t \notin (-2, 2)$,
- and $0 \le \eta(t) \le 1$ for all $t \in \mathbb{R}$.

We will call such functions *bump functions* or *cutoff functions*. We wish to have the time-localised linear solution $\eta(t)S(t)u_0$ in $X^{s,b}$, and also have its norm be comparable to the norm of u_0 in an appropriate function space. Indeed, that is the case.

Proposition 5.1.1 (Homogeneous linear estimate)**.** *Let $s, b \in \mathbb{R}$. We have*

$$\|\eta(t)S(t)u_0\|_{X^{s,b}} \lesssim_{\eta,b} \|u_0\|_{H^s_x} \tag{5.1.1}$$

for all $u_0 \in H^s_x$.

Proof. Let $u := \eta(t)S(t)u_0$. We repeat the computation in (5.0.3). See that

$$\begin{aligned}
\widehat{u}(\tau, n) &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} \eta(t)S(t)u_0(t,x)\, e^{-ix\cdot n}\, dx\, e^{-it\tau}\, dt \\
&= \int_{\mathbb{R}} \eta(t)\, e^{-it|n|^2}\, \widehat{u_0}(n)\, e^{-it\tau}\, dt \\
&= \int_{\mathbb{R}} \eta(t)\, e^{-it(\tau+|n|^2)}\, dt\, \widehat{u_0}(n) \\
&= \widehat{\eta}(\tau + |n|^2)\, \widehat{u_0}(n).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|\eta(t)S(t)u_0\|_{X^{s,b}} &= \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b\, \widehat{\eta}(\tau + |n|^2)\, \widehat{u_0}(n)\|_{\ell^2_n L^2_\tau} \\
&= \left\| \|\langle \tau + |n|^2\rangle^b\, \widehat{\eta}(\tau + |n|^2)\|_{L^2_\tau} \langle n\rangle^s\, \widehat{u_0}(n) \right\|_{\ell^2_n} \\
&= \left\| \|\eta\|_{H^b_t} \langle n\rangle^s\, \widehat{u_0}(n) \right\|_{\ell^2_n} \\
&= \|\eta\|_{H^b_t}\, \|u_0\|_{H^s_x} \\
&\lesssim_{\eta,b} \|u_0\|_{H^s_x},
\end{aligned}$$

where we used the translation invariance of the H^b_t norm. □

Naturally, we also consider time localisation for the nonhomogeneous linear estimate.

Proposition 5.1.2 (Nonhomogeneous linear estimate)**.** *Let $\varepsilon > 0$ be given, and let $s \in \mathbb{R}$ and $b > \frac{1}{2}$. Suppose that $F \in X^{s,b-1}$ and that η is a smooth cutoff function. Then, we have*

$$\|\eta\left(\frac{t}{T}\right) \int_0^t S(t-t')F(t')\, dt'\|_{X^{s,b}} \lesssim T^\varepsilon \|F\|_{X^{s,b-1+\varepsilon}} \tag{5.1.2}$$

for all $0 < T \leq 1$.

Notice that here we are not time-localising with just $\eta(t)$, but with a narrower $\eta(\frac{t}{T})$, since this allows us to get the T^ε factor in the inequality by an ε loss in regularity. We expect such a result due to the uncertainty principle.

Proof (of Proposition 5.1.2). Let $\mathcal{I}(t)$ denote the integral $\int_0^t S(t-t')F(t')\, dt'$, and given that $0 < T \leq 1$, let $\eta_T(t) = \eta(\frac{t}{T})$. Assume temporarily that

$$\|\langle \tau + |n|^2\rangle^b\, \widehat{\eta_T(t)\mathcal{I}}(\tau, n)\|_{L^2_\tau} \lesssim T^\varepsilon\, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \widehat{F}(\tau', n)\|_{L^2_{\tau'}} \tag{5.1.3}$$

for all n. Multiplying by $\langle n\rangle^s$ and taking an ℓ^2_n norm, we have

$$\begin{aligned}\|\eta_T(t)\,\mathcal{I}(t)\|_{X^{s,b}} &= \Big\| \langle n\rangle^s \, \|\langle \tau + |n|^2\rangle^b \widehat{\eta_T(t)\mathcal{I}}(\tau,n)\|_{L^2_\tau} \Big\|_{\ell^2_n} \\ &\lesssim \Big\| \langle n\rangle^s \, T^\varepsilon \, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \widehat{F}(\tau',n)\|_{L^2_{\tau'}} \Big\|_{\ell^2_n} \\ &= T^\varepsilon \, \|F\|_{X^{s,b-1+\varepsilon}},\end{aligned}$$

which is the desired result. Thus, it suffices to show that (5.1.3) holds for all n.

Let G denote the interaction representation of $\mathcal{I}$. That is,

$$G(t) = S(-t)\mathcal{I}(t) = \int_0^t S(-t')F(t')\,dt'.$$

Notice that we have

$$\begin{aligned}\mathcal{F}_x(G(t))(n) &= \int_0^t e^{it'|n|^2}\,\widehat{F}(t',n)\,dt' \\ &= \int_{\mathbb{R}} \mathbb{1}_{[0,t]}(t')\,\widehat{F}(t',n)\,e^{it'|n|^2}\,dt'.\end{aligned}$$

See that $\mathcal{F}_{t'}(\mathbb{1}_{[0,t]}(\cdot)\,\widehat{F}(\,\cdot\,,n))(\tau') = (\widehat{\mathbb{1}}_{[0,t]} *_{\tau'} \widehat{F})(\tau',n)$, where $*_{\tau'}$ denotes convolution in the τ' variable. Furthermore, the above integral is simply the Fourier transform of $\mathbb{1}_{[0,t]}(\cdot)\,\widehat{F}(\,\cdot\,,n)$ in time, evaluated at $\tau' = -|n|^2$. So, we may write the above as

$$\mathcal{F}_x(G(t))(n) = (\widehat{\mathbb{1}}_{[0,t]} *_{\tau'} \widehat{F})(\,\cdot\,,n)\big|_{\tau'=-|n|^2}.$$

Noting that

$$\widehat{\mathbb{1}}_{[0,t]}(t') = \int_0^t e^{-it'\tau'}\,dt' = \frac{e^{-it\tau'}-1}{-i\tau'},$$

we find

$$\begin{aligned}\mathcal{F}_{t,x}(\eta_T(t)\,\mathcal{I})(\tau,n) &= \mathcal{F}_{t,x}(\eta_T(t)\,S(t)G(t))(\tau,n) \\ &= \int_{\mathbb{R}} \eta_T(t) \left(e^{-it|n|^2}\,(\widehat{\mathbb{1}}_{[0,t]} *_{\tau'} \widehat{F})(\,\cdot\,,n)\big|_{\tau'=-|n|^2} \right) e^{-it\tau}\,dt \\ &= \int_{\mathbb{R}} e^{-it(\tau+|n|^2)}\,\eta_T(t) \int_{\mathbb{R}} \frac{e^{-it(-|n|^2-\tau')}-1}{-i(-|n|^2-\tau')}\,\widehat{F}(\tau',n)\,d\tau' dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\eta_T(t)\,(e^{-it(\tau-\tau')} - e^{-it(\tau+|n|^2)})}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau' dt.\end{aligned}$$

Using Fubini's theorem to switch the order of integration, we can write the above

as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_T(t)\,(e^{-it(\tau-\tau')}-e^{-it(\tau+|n|^2)})\,dt\ \frac{1}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau'$$
$$=\int_{\mathbb{R}}(\widehat{\eta_T}(\tau-\tau')-\widehat{\eta_T}(\tau+|n|^2))\ \frac{1}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau'. \tag{5.1.4}$$

See that we have

$$\int_{\mathbb{R}}(\widehat{\eta_T}(\tau-\tau')-\widehat{\eta_T}(\tau+|n|^2))\ \frac{1}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau' \le \widehat{F_1}(\tau,n)+\widehat{F_2}(\tau,n),$$

where

$$\widehat{F_1}(\tau,n) := \int_{\langle\tau'+|n|^2\rangle\lesssim\frac{1}{T}}(\widehat{\eta_T}(\tau-\tau')-\widehat{\eta_T}(\tau+|n|^2))\ \frac{1}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau',$$
$$\widehat{F_2}(\tau,n) := \int_{\langle\tau'+|n|^2\rangle\gg\frac{1}{T}}(\widehat{\eta_T}(\tau-\tau')-\widehat{\eta_T}(\tau+|n|^2))\ \frac{1}{i(\tau'+|n|^2)}\,\widehat{F}(\tau',n)\,d\tau'.$$

Fix n. We have by the triangle inequality that

$$\|\langle\tau+|n|^2\rangle^b\ \widehat{\eta_T(t)\mathcal{I}}(\tau,n)\|_{L^2_\tau} \lesssim \|\langle\tau+|n|^2\rangle^b\ \widehat{F_1}(\tau,n)\|_{L^2_\tau}+\|\langle\tau+|n|^2\rangle^b\ \widehat{F_2}(\tau,n)\|_{L^2_\tau}. \tag{5.1.5}$$

We bound each term individually:

- **Case 1:** $\langle\tau'+|n|^2\rangle\lesssim\frac{1}{T}$.

Since η_T is a Schwartz function, its Fourier transform $\widehat{\eta_T}$ is smooth. So, we may apply the Fundamental Theorem of Calculus to see that

$$\widehat{\eta_T}(\tau-\tau')-\widehat{\eta_T}(\tau+|n|^2) = -T\int_0^1 (\widehat{\eta})'(T(\tau+|n|^2)-\theta T(\tau'+|n|^2))d\theta\cdot T(\tau'+|n|^2). \tag{5.1.6}$$

Note that the $(\tau'+|n|^2)$ factor in (5.1.6) cancels out with its reciprocal in (5.1.4). Using (5.1.4), (5.1.6) and Minkowski's integral inequality, we have

$$\|\langle\tau+|n|^2\rangle^b\ \widehat{F_1}(\tau,n)\|_{L^2_\tau} \le \int_{\langle\tau'+|n|^2\rangle\lesssim\frac{1}{T}}\int_0^1 T^2A\,d\theta\ \widehat{F}(\tau',n)\,d\tau', \tag{5.1.7}$$

where

$$A := \|\langle\tau+|n|^2\rangle^b\,(\widehat{\eta})'(T(\tau+|n|^2)-\theta T(\tau'+|n|^2))\|_{L^2_\tau}.$$

We do a change of variables in A with $\tau_1 := T(\tau+|n|^2)$. So, we have

$$A = T^{-\frac{1}{2}}\,\|\langle T^{-1}\tau_1\rangle^b\,(\widehat{\eta})'(\tau_1-\theta T(\tau'+|n|^2))\|_{L^2_{\tau_1}}$$
$$\le T^{-\frac{1}{2}-b}\,\|\langle\tau_1\rangle^b\,(\widehat{\eta})'(\tau_1-\theta T(\tau'+|n|^2))\|_{L^2_{\tau_1}}. \tag{5.1.8}$$

We do a second change of variables with $\tau_2 := \tau_1-\theta T(\tau'+|n|^2)$. As $T(\tau'+|n|^2)\lesssim 1$,

we have that $\langle \tau_1 \rangle = \langle \tau_2 + \theta T(\tau' + |n|^2) \rangle \lesssim \langle \tau_2 \rangle$, and so

$$\begin{aligned} A &\leq T^{-\frac{1}{2}-b} \left\| \langle \tau_2 + \theta T(\tau' + |n|^2) \rangle^b \, (\widehat{\eta})'(\tau_2) \right\|_{L^2_{\tau_2}} \\ &\lesssim T^{-\frac{1}{2}-b} \left\| \langle \tau_2 \rangle^b \, (\widehat{\eta})'(\tau_2) \right\|_{L^2_{\tau_2}} \\ &= T^{-\frac{1}{2}-b} \left\| \langle \tau \rangle^b \, (\widehat{\eta})'(\tau) \right\|_{L^2_{\tau}} \end{aligned}$$

where in the last step we simply renamed τ_2 as τ for brevity. Thus we have from (5.1.7) that

$$\begin{aligned} \left\| \langle \tau + |n|^2 \rangle^b \, \widehat{F_1}(\tau, n) \right\|_{L^2_\tau} &\lesssim \int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \int_0^1 T^{\frac{3}{2}-b} \left\| \langle \tau \rangle^b \, (\widehat{\eta})'(\tau) \right\|_{L^2_\tau} d\theta \; \widehat{F}(\tau', n) \, d\tau' \\ &= T^{\frac{3}{2}-b} \left\| \langle \tau \rangle^b \, (\widehat{\eta})'(\tau) \right\|_{L^2_\tau} \cdot \int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \widehat{F}(\tau', n) \, d\tau' \\ &\lesssim_\eta T^{\frac{3}{2}-b} \int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \widehat{F}(\tau', n) \, d\tau'. \end{aligned} \tag{5.1.9}$$

See that we have

$$\begin{aligned} &\int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \widehat{F}(\tau', n) \, d\tau' \\ &\qquad = \int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \langle \tau' + |n|^2 \rangle^{-b+1-\varepsilon} \langle \tau' + |n|^2 \rangle^{b-1+\varepsilon} \widehat{F}(\tau', n) \, d\tau' \\ &\qquad \leq \left(\int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \langle \tau' + |n|^2 \rangle^{-2b+2-2\varepsilon} \, d\tau' \right)^{\frac{1}{2}} \left\| \langle \tau' + |n|^2 \rangle^{b-1+\varepsilon} \widehat{F}(\tau', n) \right\|_{L^2_{\tau'}} \\ &\qquad \lesssim T^{b-\frac{3}{2}+\varepsilon} \left\| \langle \tau' + |n|^2 \rangle^{b-1+\varepsilon} \widehat{F}(\tau', n) \right\|_{L^2_{\tau'}}, \end{aligned} \tag{5.1.10}$$

which follows from the Cauchy–Schwarz inequality and the fact that

$$\begin{aligned} \left(\int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} \langle \tau' + |n|^2 \rangle^{-2b+2-2\varepsilon} \, d\tau' \right)^{\frac{1}{2}} &\lesssim \left(\int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} T^{2b-2+2\varepsilon} \, d\tau' \right)^{\frac{1}{2}} \\ &= T^{b-1+\varepsilon} \left(\int_{\langle \tau' + |n|^2 \rangle \lesssim \frac{1}{T}} 1 \, d\tau' \right)^{\frac{1}{2}} \\ &\lesssim T^{b-1+\varepsilon} \, T^{-\frac{1}{2}} \\ &= T^{b-\frac{3}{2}+\varepsilon}. \end{aligned}$$

So, (5.1.9) and (5.1.10) give

$$\left\| \langle \tau + |n|^2 \rangle^b \, \widehat{F_1}(\tau, n) \right\|_{L^2_\tau} \lesssim T^\varepsilon \left\| \langle \tau' + |n|^2 \rangle^{b-1+\varepsilon} \widehat{F}(\tau', n) \right\|_{L^2_{\tau'}}. \tag{5.1.11}$$

- **Case 2:** $\langle \tau' + |n|^2 \rangle \gg \frac{1}{T}$.

We split the right-hand side of (5.1.4) into two terms

$$\int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} (\widehat{\eta_T}(\tau-\tau') - \widehat{\eta_T}(\tau+|n|^2)) \, \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau' = A_1 + A_2,$$

where

$$A_1 := \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \widehat{\eta_T}(\tau-\tau') \, \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau', \quad \text{and}$$
$$A_2 := -\widehat{\eta_T}(\tau+|n|^2) \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau'.$$

To obtain the bound (5.1.3), we are interested in controlling

$$\begin{aligned} \|\langle \tau+|n|^2\rangle^b \, \widehat{F_2}(\tau, n)\|_{L^2_\tau} &= \|\langle \tau+|n|^2\rangle^b \, (A_1 + A_2)\|_{L^2_\tau} \\ &\le \|\langle \tau+|n|^2\rangle^b \, A_1\|_{L^2_\tau} + \|\langle \tau+|n|^2\rangle^b \, A_2\|_{L^2_\tau}, \end{aligned} \tag{5.1.12}$$

where we used the triangle inequality. We investigate each of the two norms on the right-hand side individually.

For the first norm, see that the triangle inequality implies

$$\langle \tau+|n|^2\rangle^b \lesssim \langle \tau-\tau'\rangle^b + \langle \tau'+|n|^2\rangle^b,$$

which means that

$$\|\langle \tau+|n|^2\rangle^b \, A_1\|_{L^2_\tau} \lesssim \|A_{1,1}\|_{L^2_\tau} + \|A_{1,2}\|_{L^2_\tau} \tag{5.1.13}$$

with

$$A_{1,1} := \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \langle \tau-\tau'\rangle^b \, \widehat{\eta_T}(\tau-\tau') \, \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau'$$
$$A_{1,2} := \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \langle \tau'+|n|^2\rangle^b \, \widehat{\eta_T}(\tau-\tau') \, \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau'$$

We have

$$\|A_{1,1}\|_{L^2_\tau} \le T^{\frac{1}{2}-b} \, \|\langle \tau\rangle^b \, \widehat{\eta}(\tau)\|_{L^2_\tau} \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau'$$

by Minkowski's integral inequality and a computation similar to (5.1.9). Here, the $T^{\frac{1}{2}-b}$ factor appears as a result of a change of variables argument similar to (5.1.8). Moreover, arguing similarly to (5.1.10), we get

$$\int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \frac{1}{i(\tau'+|n|^2)} \, \widehat{F}(\tau', n)\, d\tau' \lesssim T^{b-\frac{1}{2}+\varepsilon} \|\langle \tau'+|n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}},$$

and so we have

$$\|A_{1,1}\|_{L^2_\tau} \lesssim_\eta T^\varepsilon \, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}}. \tag{5.1.14}$$

Furthermore, we have

$$\|A_{1,2}\|_{L^2_\tau} \lesssim \| \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} |\widehat{\eta}(\tau-\tau')| \, \frac{|\widehat{F}(\tau', n)|}{\langle \tau' + |n|^2\rangle^{1-b-\varepsilon}} \, \langle \tau' + |n|^2\rangle^{-\varepsilon} \, d\tau' \|_{L^2_\tau}.$$

This is the L^2_τ norm of a convolution on the variable τ'. Hence, we apply Young's convolution inequality to get

$$\begin{aligned} \|A_{1,2}\|_{L^2_\tau} &\lesssim \|\widehat{\eta_T}\|_{L^1_\tau} \, \| \frac{\widehat{F}(\tau', n)}{\langle \tau' + |n|^2\rangle^{1-b-\varepsilon}} \, \langle \tau' + |n|^2\rangle^{-\varepsilon} \|_{L^2_{\tau'}} \\ &\lesssim \|\widehat{\eta}\|_{L^1_\tau} \, T^\varepsilon \, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}} \\ &\lesssim_\eta T^\varepsilon \, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}}, \end{aligned} \tag{5.1.15}$$

where we have used $\|\widehat{\eta_T}\|_{L^1_\tau} = \|\widehat{\eta}\|_{L^1_\tau}$ (by a change of variables $\tau \to T\tau$) and the fact that $\langle \tau' + |n|^2\rangle^{-\varepsilon} \lesssim T^\varepsilon$.

We finally show that the same bound holds for the second norm on the right-hand side of (5.1.12). See that

$$\begin{aligned} &\|\langle \tau + |n|^2\rangle^b \, A_2\|_{L^2_\tau} \\ &\quad \lesssim \|\langle \tau + |n|^2\rangle^b \, \widehat{\eta_T}(\tau + |n|^2) \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \frac{1}{i(\tau' + |n|^2)} \, \widehat{F}(\tau', n) \, d\tau' \|_{L^2_\tau} \\ &\quad \lesssim \|\langle \tau + |n|^2\rangle^b \, \widehat{\eta_T}(\tau + |n|^2)\|_{L^2_\tau} \int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \frac{\langle \tau' + |n|^2\rangle^{b-1+\varepsilon}}{\langle \tau' + |n|^2\rangle^{b-\varepsilon}} \, |\widehat{F}(\tau', n)| \, d\tau' \\ &\quad \lesssim_\eta T^{\frac{1}{2}-b} \Big(\int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \langle \tau' + |n|^2\rangle^{-2b+2\varepsilon} \, d\tau' \Big)^{\frac{1}{2}} \, \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}} \\ &\quad \lesssim T^\varepsilon \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}}, \end{aligned} \tag{5.1.16}$$

where we have used

$$\|\langle \tau + |n|^2\rangle^b \, \widehat{\eta_T}(\tau + |n|^2)\|_{L^2_\tau} \lesssim_\eta T^{\frac{1}{2}-b}$$

by the usual change of variables argument, and

$$\Big(\int_{\langle \tau'+|n|^2\rangle \gg \frac{1}{T}} \langle \tau' + |n|^2\rangle^{-2b+2\varepsilon} \, d\tau' \Big)^{\frac{1}{2}} \lesssim T^{b-\frac{1}{2}+\varepsilon},$$

which follows from $\langle \tau' + |n|^2\rangle \gg \frac{1}{T}$. Combining (5.1.12), (5.1.13), (5.1.14), (5.1.15), and (5.1.16), we get

$$\|\langle \tau + |n|^2\rangle^b \, \widehat{F_2}(\tau, n)\|_{L^2_\tau} \lesssim T^\varepsilon \|\langle \tau' + |n|^2\rangle^{b-1+\varepsilon} \, \widehat{F}(\tau', n)\|_{L^2_{\tau'}}. \tag{5.1.17}$$

Finally, (5.1.5), (5.1.11) and (5.1.17) imply (5.1.3), which is the desired bound. □

5.2 Transference principle

Recall that we are considering nonlinearities of the form $\mathcal{N}(u) = \pm|u|^2u$, which are products of u and $\overline{u}$. In particular, we will be considering the cubic NLS on $\mathbb{T}$ in Chapter 6 and on $\mathbb{T}^2$ in Chapter 7. To prove the relevant nonlinear estimates, we require some way of transferring Strichartz estimates to estimates in $X^{s,b}$ spaces.

Lemma 5.2.1 (Transference principle)**.** *Let $b > \frac{1}{2}$, and let $\mathcal{T}$ be a k-linear operator such that*

$$\|\mathcal{T}(S(t)f_1, \dots, S(t)f_k)\|_{L^q_t L^r_x} \lesssim \prod_{j=1}^{k} \|f_j\|_{H^{s_j}}$$

for all $f_j \in H^{s_j}$. Then, we have

$$\|\mathcal{T}(u_1, \dots, u_k)\|_{L^q_t L^r_x} \lesssim \prod_{j=1}^{k} \|u_j\|_{X^{s_j,b}}$$

for all $u_j \in X^{s_j,b}$.

Proof. Let $u_j \in X^{s_j,b}$ for $1 \le j \le k$. We set

$$f_{j,\tau_j}(x) := \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau_j - |n|^2, n)\, e^{ix\cdot n}.$$

We have

$$S(t)f_{j,\tau_j}(x) = \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau_j - |n|^2, n)\, e^{ix\cdot n - it|n|^2},$$

which means that

$$\begin{aligned}
\int_{\mathbb{R}} S(t)f_{j,\tau_j}(x)\, e^{it\tau_j}\, d\tau_j &= \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}^d} \widehat{u_j}(\tau_j - |n|^2, n)\, e^{ix\cdot n}\, e^{it(\tau_j - |n|^2)}\, d\tau_j \\
&= \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}} \widehat{u_j}(\tau_j - |n|^2, n)\, e^{it(\tau_j - |n|^2)}\, d\tau_j\, e^{ix\cdot n} \\
&= u_j, \qquad (5.2.1)
\end{aligned}$$

where we have used Fubini's theorem and a change of variables $\tau := \tau_j - |n|^2$.

Moreover, using Cauchy–Schwarz, we also have that

$$\begin{aligned}\int_{\mathbb{R}} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_j &= \int_{\mathbb{R}} \langle\tau_j\rangle^{-b}\, \langle\tau_j\rangle^{b} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_j \\ &\leq \left(\int_{\mathbb{R}} \langle\tau_j\rangle^{-2b} d\tau_j\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle\tau_j\rangle^{2b} \|f_{j,\tau_j}\|_{H^{s_j}}^2\, d\tau_j\right)^{\frac{1}{2}} \\ &\lesssim_b \left\| \langle\tau_j\rangle^{b}\, \|f_{j,\tau_j}\|_{H^{s_j}} \right\|_{L^2_{\tau_j}},\end{aligned}$$

which follows from the integral $\int_{\mathbb{R}} \langle\tau_j\rangle^{-2b} d\tau_j$ being finite for $b > \frac{1}{2}$. Using the definition of f_{j,τ_j}, and exchanging the order of norms using Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_j &\lesssim \left\| \langle\tau_j\rangle^{b}\, \|\langle n\rangle^{s_j}\, \widehat{u_j}(\tau_j - |n|^2, n)\|_{\ell^2_n} \right\|_{L^2_{\tau_j}} \\ &= \left\| \langle n\rangle^{s_j}\, \|\langle\tau_j\rangle^{b}\, \widehat{u_j}(\tau_j - |n|^2, n)\|_{L^2_{\tau_j}} \right\|_{\ell^2_n} = \|u_j\|_{X^{s,b}} \qquad (5.2.2)\end{aligned}$$

by a change of variables. In particular, $\|f_{j,\tau_j}\|_{H^{s_j}}$ is finite for almost every τ_j since $\|u_j\|_{X^{s,b}}$ is finite, meaning that $f_{j,\tau_j} \in H^{s_j}$ for almost every τ_j.

Then, we have from (5.2.1) and the k-linearity and boundedness of $\mathcal{T}$ that

$$\begin{aligned}\mathcal{T}(u_1, \dots, u_k) &= \mathcal{T}\Big(\int_{\mathbb{R}} S(t) f_{1,\tau_1}(x)\, e^{it\tau_1}\, d\tau_1, \dots, \int_{\mathbb{R}} S(t) f_{k,\tau_k}(x)\, e^{it\tau_k}\, d\tau_k\Big) \\ &= \int_{\mathbb{R}^k} e^{it\tau_1} \dots e^{it\tau_k}\, \mathcal{T}(S(t) f_{1,\tau_1}, \dots, S(t) f_{k,\tau_k})\, d\tau_1 \dots d\tau_k.\end{aligned}$$

So, by Minkowski's integral inequality, the $L^q_t L^r_x$ norm of the above is bounded by

$$\begin{aligned}\|\mathcal{T}(u_1, \dots, u_k)\|_{L^q_t L^r_x} &\lesssim \int_{\mathbb{R}^k} |e^{it\tau_1} \dots e^{it\tau_k}|\, \|\mathcal{T}(S(t) f_{1,\tau_1}, \dots, S(t) f_{k,\tau_k})\|_{L^q_t L^r_x} d\tau_1 \dots d\tau_k \\ &\lesssim \int_{\mathbb{R}^k} \prod_{j=1}^{k} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_1 \dots d\tau_k \\ &= \prod_{j=1}^{k} \int_{\mathbb{R}} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_j\end{aligned}$$

using the assumed bound on the operator $\mathcal{T}$. By (5.2.2), we have

$$\|\mathcal{T}(u_1, \dots, u_k)\|_{L^q_t L^r_x} \lesssim \prod_{j=1}^{k} \int_{\mathbb{R}} \|f_{j,\tau_j}\|_{H^{s_j}}\, d\tau_j \lesssim \prod_{j=1}^{k} \|u_j\|_{X^{s,b}},$$

which is the desired result. □

The underlying principle in the above proof is that for $b > \frac{1}{2}$, we may consider

$u \in X^{s,b}$ to be a superposition of linear solutions, see (5.2.1). Notice also that this is the same condition for the embedding $X^{s,b} \subset C_T H^s_x$ in Lemma 5.0.3 and the nonhomogeneous linear estimate (5.1.2).

5.3 An alternative definition

We conclude this chapter with a remark on an alternative way of defining the Fourier restriction norm spaces $X^{s,b}$. As we have seen throughout this chapter, most fundamental properties and results concerning $X^{s,b}$ spaces require time localisation using some cutoff function η. On the other hand, this cutoff function is somewhat arbitrary. So, results such as Propositions 5.1.1 and 5.1.2 are true for any cutoff function η. With this observation, given $T > 0$, we may define local-in-time $X^{s,b}(T)$ spaces using the norm

$$\|u\|_{X^{s,b}(T)} := \inf\{\|v\|_{X^{s,b}} : v\big|_{[-T,T]} = u\},$$

where $\|\cdot\|_{X^{s,b}}$ is defined in the usual way, and as such we may simply omit cutoff functions in our estimates under this $X^{s,b}(T)$ norm. Here, we are denoting by v an extension of u, who is defined on $[-T, T]$, onto $\mathbb{R}$. We will, however, choose to continue using the norm as given in Defintion 5.0.1, and so we will be explicit whenever we require a cutoff function.

Chapter 6

Cubic nonlinear Schrödinger equation on $\mathbb{T}$

We wish to deal with the nonlinearity, using an approach similar to what we discussed in Chapter 4. To that aim, we first establish a Strichartz estimate.

6.1 Strichartz estimate

To obtain a nonlinear estimate, we hope to employ a simple duality argument to pass from an $X^{s,b}$ norm to an L^1 norm, and apply Hölder's inequality as we did in Chapter 4.2 in several ways. Then, we require a Strichartz-type estimate to allow us to pass back into the right $X^{s,b}$ space.

The proofs for Strichartz estimates on $\mathbb{T}^d$ that we consider involve using a counting argument to restrict summations, as was the motivation for defining the Fourier restriction norm spaces in Definition 5.0.1. As to introduce this type of argument and to remark on a related result, we consider the following one dimensional L^4 estimate in periodic time first proven by Zygmund [37]:

Proposition 6.1.1 (One dimensional L^4 Strichartz estimate in periodic time [37])**.** *The following estimate holds*

$$\|S(t)f\|_{L^4_{t,x}(\mathbb{T}_t\times\mathbb{T}_x)} \lesssim \|f\|_{L^2_x(\mathbb{T})}. \tag{6.1.1}$$

Proof. Let

$$u(t,x) := S(t)f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{i(nx-tn^2)}.$$

Then, we have

$$\|u\|^2_{L^4_{t,x}} = \|u\overline{u}\|_{L^2_{t,x}} = \| \sum_{n_1,n_2\in\mathbb{Z}} \widehat{f}(n_1)\,\overline{\widehat{f}(n_2)}\, e^{-it(n_1^2-n_2^2)}\, e^{i(n_1-n_2)x}\|_{L^2_{t,x}}.$$

The sum on the right-hand side is thus a Fourier series sum in (t, x) of the form

$$u\overline{u} = \sum_{(\tau,n)\in\mathbb{Z}^2} a(\tau, n)\, e^{i(nx+t\tau)}, \tag{6.1.2}$$

where

$$a(\tau, n) := \sum_{(n_1,n_2)\in P(\tau,n)} \widehat{f}(n_1)\,\overline{\widehat{f}(n_2)}$$

and

$$P(\tau, n) := \{(n_1, n_2) \in \mathbb{Z}^2 : n_1^2 - n_2^2 = -\tau,\ n_1 - n_2 = n\}.$$

Given some $(\tau, n) \neq (0, 0)$, there exists at most one choice of $(n_1, n_2) \in \mathbb{Z}^2$ such that the equations

$$\begin{aligned} \tau &= -n_1^2 + n_2^2, \\ n &= n_1 - n_2 \end{aligned}$$

are satisfied. This is because given the above, assuming $n \neq 0$ we have

$$\tau + n^2 = -n_1^2 + n_2^2 + (n_1 - n_2)^2 = -2n_2(n_1 - n_2) = -2n_2 n,$$

and so $n_2 = \frac{\tau+n^2}{-n}$ and $n_1 = n - n_2$. Moreover, if $n = 0$, then $\tau \neq 0$ and no solution exists since $\tau = -n_1^2 + n_2^2 = -(n_1 - n_2)(n_1 + n_2) = 0$ gives a contradiction. Consequently, we have

$$\begin{aligned} \sum_{(\tau,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} |a(n_1, n_2)|^2 &\lesssim \sum_{n_1,n_2\in\mathbb{Z}} |\widehat{f}(n_1)\,\overline{\widehat{f}(n_2)}|^2 \\ &= \sum_{n_1\in\mathbb{Z}} |\widehat{f}(n_1)|^2 \sum_{n_2\in\mathbb{Z}} |\overline{\widehat{f}(n_2)}|^2 \\ &= \|f\overline{f}\|_{L_x^2}^2 \\ &= \|f\|_{L_x^2}^4, \end{aligned} \tag{6.1.3}$$

where we have used Plancherel's identity.

On the other hand, if $(\tau, n) = (0, 0)$, we have $(n_1, n_2) \in P(0, 0)$ if and only if $n_1 = n_2$, and so

$$a(0, 0) = \sum_{(n_1,n_2)\in P(0,0)} \widehat{f}(n_1)\,\overline{\widehat{f}(n_2)} = \sum_{n_1\in\mathbb{Z}} |\widehat{f}(n_1)|^2 = \|f\|_{L_x^2}^2 \tag{6.1.4}$$

by Plancherel's identity.

Therefore, using (6.1.2), Plancherel's identity, (6.1.3), and (6.1.4), we have

$$\|u\|_{L_{t,x}^2(\mathbb{T}_t,\mathbb{T}_x)}^4 = \|u\overline{u}\|_{L_{t,x}^2}^2$$

$$= |a(0,0)|^2 + \sum_{(\tau,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} |a(n_1,n_2)|^2$$
$$\lesssim \|f\|_{L^2_x}^2,$$

which directly implies the desired bound (6.1.1). □

Unfortunately, this result is insufficient to conduct a duality argument for the nonlinear estimate. Recall Lemma 5.0.2. Using the transference principle (Lemma 5.2.1) with (6.1.1), we get the bound

$$\|\eta(t)\, u\|_{L^4_{t,x}(\mathbb{R}_t\times\mathbb{T}_x)} \lesssim \|u\|_{X^{0,b}},$$

which holds for $b > \frac{1}{2}$. Since the nonhomogeneous linear estimate 5.1.2 ends up at the space $X^{0,-\frac{1}{2}+2\varepsilon}$, duality would take us to $X^{0,\frac{1}{2}-2\varepsilon}$. This means that we require using a space where $0 < b < \frac{1}{2}$ for the duality argument, which is strictly outside of the range permitted by the above bound. On the other hand, notice that from the Sobolev inequality, we have

$$\|u\|_{L^4_{t,x}(\mathbb{R}_t\times\mathbb{T}_x)} \lesssim \|u\|_{X^{\frac{1}{4},\frac{1}{4}}},$$

and so by the heuristics discussed earlier (5.0.5), we should expect a bound in $X^{0,\frac{3}{8}}$. Roughly twenty years after Zygmund's result, this necessary improvement came in Bourgain's 1993 paper [3]:

Proposition 6.1.2 (One dimensional L^4 Strichartz estimate [3])**.** *Let $u \in X^{0,\frac{3}{8}}$. The following estimate holds*

$$\|u\|_{L^4_{t,x}(\mathbb{R}_t\times\mathbb{T}_x)} \lesssim \|u\|_{X^{0,\frac{3}{8}}}. \tag{6.1.5}$$

Proof. This proof is by Nikolay Tzvetkov and can also be found in [32, Chapter 2.6]. We will apply several reductions to the problem. We do a dyadic decomposition of u, such that

$$u = \sum_M u_M,$$

where u_M for $M := 2^j$ with $j \in \mathbb{Z}_{\geq 0}$ are pieces of u localised to the space-time frequency regions where $\langle \tau + n^2\rangle \sim M$ (recall Remark 2.1.1). See that

$$\sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} = \sum_M M^{\frac{3}{4}} \|\widehat{u_M}\|^2_{\ell^2_n L^2_\tau} \leq \sum_M \|\langle \tau + |n|^2\rangle^{\frac{3}{8}} \widehat{u_M}\|^2_{\ell^2_n L^2_\tau} \sim \|u\|^2_{X^{0,\frac{3}{8}}}$$

by Plancherel's identity. Moreover, notice that by Minkowski's inequality we have

$$\|u\|^2_{L^4_{t,x}} = \|\sum_M u_M \sum_{M'} u_{M'}\|_{L^2_{t,x}} \leq \sum_M \sum_{M'} \|u_M u_{M'}\|_{L^2_{t,x}} \lesssim \sum_{M\leq M'} \|u_M u_{M'}\|_{L^2_{t,x}}.$$

This means that the desired bound (6.1.5) follows once we show

$$\sum_{M\leq M'} \|u_M u_{M'}\|_{L^2_{t,x}} \lesssim \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}}. \tag{6.1.6}$$

Notice that since both M and M' are dyadic, the condition $M \leq M'$ is satisfied if and only if $M' = 2^m M$ for some $m \in \mathbb{Z}_{\geq 0}$. Then, if we have

$$\sum_M \|u_M u_{2^m M}\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon m} \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} \tag{6.1.7}$$

for all m for some $\varepsilon > 0$, we can compute

$$\sum_M \sum_{m\geq 0} \|u_M u_{2^m M}\|_{L^2_{t,x}} \lesssim \sum_M \sum_{m\geq 0} 2^{-\varepsilon m} M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} \lesssim \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}},$$

since $\sum_{m\geq 0} 2^{-\varepsilon m} \lesssim 1$. So, (6.1.6) follows from (6.1.7). Moreover, if we show

$$\|u_M u_{2^m M}\|_{L^2_{t,x}} \lesssim 2^{-\varepsilon m} M^{\frac{3}{8}} \|u_M\|_{L^2_{t,x}} (2^m M)^{\frac{3}{8}} \|u_{2^m M}\|_{L^2_{t,x}}, \tag{6.1.8}$$

then we find

$$\begin{aligned}
\sum_M \|u_M u_{2^m M}\|_{L^2_{t,x}} &\lesssim 2^{-\varepsilon m} \sum_M M^{\frac{3}{8}} \|u_M\|_{L^2_{t,x}} (2^m M)^{\frac{3}{8}} \|u_{2^m M}\|_{L^2_{t,x}} \\
&\leq 2^{-\varepsilon m} \left(\sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} \left(\sum_M (2^m M)^{\frac{3}{4}} \|u_{2^m M}\|^2_{L^2_{t,x}} \right)^{\frac{1}{2}} \\
&\leq 2^{-\varepsilon m} \sum_M M^{\frac{3}{4}} \|u_M\|^2_{L^2_{t,x}}
\end{aligned}$$

using Cauchy–Schwarz, so (6.1.8) implies (6.1.7). We normalise u_M and $u_{2^m M}$ to define

$$v_M := \frac{u_M}{\|u_M\|_{L^2_{t,x}}}, \quad \text{and} \quad v_{2^m M} := \frac{u_{2^m M}}{\|u_{2^m M}\|_{L^2_{t,x}}}.$$

Then, we can write (6.1.8) as

$$\|v_M v_{2^m M}\|_{L^2_{t,x}} \lesssim 2^{(\frac{3}{8}-\varepsilon)m} M^{\frac{3}{4}},$$

and by Plancherel's identity, it suffices to show

$$\| \sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} \widehat{v_M}(\tau_1, n_1)\, \widehat{v_{2^m M}}(\tau_2, n_2)\, d\tau_1 \|_{\ell^2_n L^2_\tau} \lesssim 2^{(\frac{3}{8}-\varepsilon)m} M^{\frac{3}{4}}. \tag{6.1.9}$$

Let

$$A(\tau, n) := \left(\sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} |\widehat{v_M}(\tau_1, n_1)|^2\, |\widehat{v_{2^m M}}(\tau_2, n_2)|^2\, d\tau_1 \right)^{\frac{1}{2}},$$

and see that

$$\begin{aligned}\|A\|_{\ell^2_n L^2_\tau} &= \Big(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} |\widehat{v_M}(\tau_1,n_1)|^2\, |\widehat{v_{2^m M}}(\tau_2,n_2)|^2\, d\tau_1\, d\tau\Big)^{\frac12}\\ &= \Big(\sum_{n_1,n_2\in\mathbb{Z}} \int_{\mathbb{R}^2} |\widehat{v_M}(\tau_1,n_1)|^2\, |\widehat{v_{2^m M}}(\tau_2,n_2)|^2\, d\tau_1\, d\tau_2\Big)^{\frac12}\\ &= \Big(\sum_{n_1\in\mathbb{Z}} \int_{\mathbb{R}} |\widehat{v_M}(\tau_1,n_1)|^2\, d\tau_1\Big)^{\frac12} \Big(\sum_{n_2\in\mathbb{Z}} \int_{\mathbb{R}} |\widehat{v_{2^m M}}(\tau_2,n_2)|^2\, d\tau_2\Big)^{\frac12}\\ &= \|v_M\|_{\ell^2_n L^2_\tau}\, \|v_{2^m M}\|_{\ell^2_n L^2_\tau} = 1,\end{aligned}$$

where we have used Fubini's theorem. Now, recall that $\widehat{v_M}(\tau_1,n_1)$ and $\widehat{v_{2^m M}}(\tau_2,n_2)$ are nonzero only on a dyadic region around M and $2^m M$, respectively. So, let

$$\begin{aligned}R_1 &:= \{(\tau_1,n_1)\in\mathbb{R}\times\mathbb{T} : \langle \tau_1 + {n_1}^2\rangle \sim M\},\\ R_2 &:= \{(\tau_2,n_2)\in\mathbb{R}\times\mathbb{T} : \langle \tau_2 + {n_2}^2\rangle \sim 2^m M\}.\end{aligned}$$

By Cauchy–Schwarz, write the left-hand side of (6.1.9) as

$$\begin{aligned}&\Big\| \sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} \mathbb{1}_{R_1}(\tau_1,n_1)\, \mathbb{1}_{R_2}(\tau_2,n_2)\, \widehat{v_M}(\tau_1,n_1)\, \widehat{v_{2^m M}}(\tau_2,n_2)\, d\tau_1 \Big\|_{\ell^2_n L^2_\tau}\\ &\qquad = \Big\| \Big(\sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} \mathbb{1}_{R_1}(\tau_1,n_1)\, \mathbb{1}_{R_2}(\tau_2,n_2)\, d\tau_1 \Big)^{\frac12} A \Big\|_{\ell^2_n L^2_\tau}\\ &\qquad = \Big\| \Big(\sum_{n_1+n_2=n} \int_{\substack{\tau_1+\tau_2=\tau\\ (\tau_1,n_1)\in R_1\\ (\tau_2,n_2)\in R_2}} 1\, d\tau_1 \Big)^{\frac12} A \Big\|_{\ell^2_n L^2_\tau}\\ &\qquad \le \sup_{\tau,n} \Big(\sum_{n_1+n_2=n} \int_{\substack{\tau_1+\tau_2=\tau\\ (\tau_1,n_1)\in R_1\\ (\tau_2,n_2)\in R_2}} 1\, d\tau_1 \Big)^{\frac12} \|A\|_{\ell^2_n L^2_\tau}.\end{aligned}$$

Since $\|A\|_{\ell^2_n L^2_\tau} = 1$, (6.1.9) follows once we show

$$\sup_{\tau,n} \sum_{n_1+n_2=n} \int_{\substack{\tau_1+\tau_2=\tau\\ (\tau_1,n_1)\in R_1\\ (\tau_2,n_2)\in R_2}} 1\, d\tau_1 \lesssim 2^{(\frac34-2\varepsilon)m} M^{\frac32}. \tag{6.1.10}$$

We have thus reduced the problem into a counting problem. Fix τ and n. See that, for a given n_1, for (τ_1,n_1) to be in the region R_1, we require $\langle \tau_1 + n_1^2\rangle$ to be less than $2M$, and so τ_1 can attain values in a range of measure $O(M)$. Similarly, τ_2 can vary by only $O(2^m M)$ for a given n_2. We can thus consider

$$\begin{cases} \tau_1+\tau_2=\tau\\ (\tau_1,n_1)\in R_1\\ (\tau_2,n_2)\in R_2 \end{cases} \implies \begin{cases} \tau_1+\tau_2=\tau\\ \tau_1+n_1^2 = O(M)\\ \tau_2+n_2^2 = O(2^m M) \end{cases}.$$

From this, we immediately see that the integral in τ_1 in (6.1.10) evaluates to $O(M)$. It remains the count the choices for (n_1, n_2) such that

$$\begin{aligned} n &= n_1 + n_2, \\ -\tau &= -\tau_1 - \tau_2 = n_1^2 + n_2^2 + O(2^m M). \end{aligned}$$

See that we have

$$\begin{aligned} (n_1 - n_2)^2 &= n_1^2 + n_2^2 - 2n_1 n_2 \\ &= 2(n_1^2 + n_2^2) - (n_1 + n_2)^2 \\ &= -2\tau - n + O(2^m M). \end{aligned}$$

Since τ and n are fixed, this means that there are $O(2^{\frac{m}{2}} M^{\frac{1}{2}})$ choices for $(n_1 - n_2)$, and consequently for n_1 and n_2 since $n = n_1 + n_2$ is fixed. All together, this implies that

$$\sup_{\tau,n} \sum_{n_1+n_2=n} \int_{\substack{\tau_1+\tau_2=\tau \\ (\tau_1,n_1)\in R_1 \\ (\tau_2,n_2)\in R_2}} 1 \, d\tau_1 \lesssim \sup_{\tau,n} 2^{\frac{m}{2}} M^{\frac{1}{2}}(M) = 2^{\frac{m}{2}} M^{\frac{3}{2}} \lesssim 2^{(\frac{3}{4}-2\varepsilon)m} M^{\frac{3}{2}},$$

for $\varepsilon \leq \frac{1}{8}$, which is the bound (6.1.10). Therefore, we have the desired result. □

6.2 Contraction mapping argument

We first prove the nonlinear estimate.

Proposition 6.2.1 (Nonlinear estimate in $\mathbb{T}$ [3]). *Let $u_1, u_2, u_3 \in X^{0,\frac{1}{2}+\varepsilon}$. We have*

$$\|u_1 \overline{u_2} u_3\|_{X^{0,-\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{0,\frac{1}{2}+\varepsilon}} \tag{6.2.1}$$

for all $0 < \varepsilon \leq \frac{1}{16}$.

Proof. We have from a duality argument that

$$\|u_1 \overline{u_2} u_3\|_{X^{0,-\frac{1}{2}+2\varepsilon}} = \sup_{\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \left| \iint_{\mathbb{R}\times\mathbb{T}} u_1 \overline{u_2} u_3 \overline{u_4} \, dx \, dt \right|.$$

Applying Hölder's inequality and the Strichartz estimate (6.1.5) we get

$$\begin{aligned} \|u_1 \overline{u_2} u_3\|_{X^{0,-\frac{1}{2}+2\varepsilon}} &\leq \sup_{\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \prod_{j=1}^{4} \|u_j\|_{L^4_{t,x}} \\ &\lesssim \sup_{\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \prod_{j=1}^{4} \|u_j\|_{X^{0,\frac{3}{8}}}. \end{aligned}$$

Let $0 < \varepsilon \leq \frac{1}{16}$. That is, $\frac{3}{8} \leq \frac{1}{2} - 2\varepsilon$, and so we have

$$\begin{aligned}\|u_1\overline{u_2}u_3\|_{X^{0,-\frac{1}{2}+2\varepsilon}} &\lesssim \sup_{\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \prod_{j=1}^{4} \|u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &= \prod_{j=1}^{3} \|u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim \prod_{j=1}^{3} \|u_j\|_{X^{0,\frac{1}{2}+\varepsilon}},\end{aligned}$$

which is the desired result. □

From here, the proof of Theorem 1.0.1 is similar to the proof of Theorem 3.0.1, and thus we omit the details. To carry out the proof, one runs a contraction mapping argument on $X^{0,\frac{1}{2}+\varepsilon}$ for small $\varepsilon > 0$, defining the solution map

$$\Gamma_{u_0}(u) := \eta(t)\, S(t)u_0 \mp i\eta\left(\frac{t}{T}\right) \int_0^t S(t-t')|u|^2 u(t')\, dt'$$

with $0 < T < 1$ whose value is to be determined. Then, the homogeneous linear 5.1.1, nonhomogenous linear 5.1.2, and nonlinear 6.2.1 estimates, and an application of Lemma 5.0.3 gives that the cubic NLS is locally well-posed in $L^2(\mathbb{T})$.

Chapter 7

Cubic nonlinear Schrödinger equation on $\mathbb{T}^2$

Here, we wish to follow a procedure identical to that of the previous chapter. We first consider the Strichartz estimate on $\mathbb{T}^2$.

7.1 Strichartz estimate

We will prove the following Strichartz estimate, which is analogous to (6.1.5) in two dimensions, though with a small loss.

Proposition 7.1.1 (Two dimensional L^4 Strichartz estimate [3])**.** *Let $\varepsilon > 0$, $N \in \mathbb{N}$ and $f \in L^2_x(\mathbb{T}^2)$. The following estimate holds*

$$\|\eta(t)\, S(t) P_{\leq N} f\|_{L^4_{t,x}(\mathbb{R}_t \times \mathbb{T}^2_x)} \lesssim N^{\varepsilon} \|P_{\leq N} f\|_{L^2_x(\mathbb{T}^2)}. \tag{7.1.1}$$

To prove this proposition, we require the following divisor counting lemma:

Lemma 7.1.2. *Let $n \neq 0$ be a Gaussian integer, and let $d(n)$ denote the number of divisors of n. We have*

$$d(n) \lesssim_{\varepsilon} |n|^{\varepsilon}$$

for all $\varepsilon > 0$.

The proof of this lemma for $n \in \mathbb{Z}$ can be found in [13, Chapter XVIII]. The same proof can be adapted for Gaussian integers $\mathbb{Z}[i]$, since $\mathbb{Z}[i]$ too is a Euclidean domain.

Proof (of Proposition 7.1.1). Let $g = P_{\leq N} f$. See that

$$\begin{aligned} \|\eta(t)\, S(t) g\|^4_{L^4_{t,x}} &= \| \left(\eta(t)\, S(t) g\right) \left(\eta(t)\, S(t) g\right) \|^2_{L^2_{t,x}} \\ &= \| \sum_{n_1 + n_2 = n} \int_{\tau_1 + \tau_2 = \tau} \prod_{j=1}^{2} \widehat{\eta}(\tau_j + |n_j|^2)\, \widehat{g}(n_j)\, d\tau_1 \|^2_{\ell^2_n L^2_\tau} \end{aligned} \tag{7.1.2}$$

by Plancherel's identity. Let $\widehat{\beta_1} := \widehat{\eta}(\,\cdot + |n_1|^2)$ and $\widehat{\beta_2} := \widehat{\eta}(\,\cdot - |n_1|^2)$. Notice that

$$\begin{aligned}\int_{\tau_1+\tau_2=\tau} \widehat{\eta}(\tau_1+|n_1|^2)\,\widehat{\eta}(\tau_2+|n_2|^2)\,d\tau_1 &= \int_{\mathbb{R}} \widehat{\beta_1}(\tau_1)\,\widehat{\beta_2}(\tau+|n_1|^2+|n_2|^2-\tau_1)\,d\tau_1\\ &= \psi(\tau+|n_1|^2+|n_2|^2),\end{aligned}$$

where $\psi := \widehat{\beta_1} * \widehat{\beta_2}$. Since η is a Schwartz function, so are β_1 and β_2. So, $\psi = \widehat{\beta_1\beta_2}$ is also a Schwartz function. We can thus write

$$\sum_{n_1+n_2=n} \int_{\tau_1+\tau_2=\tau} \prod_{j=1}^{2} \widehat{\eta}(\tau_j+|n_j|^2)\,\widehat{g}(n_j)\,d\tau_1 = \sum_{n_1+n_2=n} \psi(\tau+|n_1|^2+|n_2|^2) \prod_{j=1}^{2} \widehat{g}(n_j),$$

and so (7.1.2) becomes

$$\begin{aligned}\|\eta(t)\,S(t)g\|_{L^4_{t,x}}^4 &= \| \sum_{n_1+n_2=n} \psi(\tau+|n_1|^2+|n_2|^2) \prod_{j=1}^{2} \widehat{g}(n_j)\|_{\ell^2_n L^2_\tau}^2\\ &= \| \sum_{n_1+n_2=n} |\psi(\tau+|n_1|^2+|n_2|^2)|^2 \prod_{j=1}^{2} |\widehat{g}(n_j)|^2 \,\|_{\ell^1_n L^1_\tau}\\ &\le \| \sum_{\substack{n_1+n_2=n\\ |n_1|,|n_2|\le N}} |\psi(\tau+|n_1|^2+|n_2|^2)|\,\|_{\ell^\infty_n L^\infty_\tau}\ \|A\|_{\ell^1_n L^1_\tau}, \qquad (7.1.3)\end{aligned}$$

by Hölder's inequality, where we have the condition $|n_1|, |n_2| \le N$ in the sum due to the support of $\widehat{g}$, and

$$A := \sum_{n_1+n_2=n} |\psi(\tau+|n_1|^2+|n_2|^2)| \prod_{j=1}^{2} |\widehat{g}(n_j)|^2.$$

See that

$$\begin{aligned}\|A\|_{\ell^1_n L^1_\tau} &= \Big\| \sum_{n_1+n_2=n} \|\psi(\tau+|n_1|^2+|n_2|^2)\|_{L^1_\tau} \prod_{j=1}^{2} |\widehat{g}(n_j)|^2 \Big\|_{\ell^1_n}\\ &= \|\psi\|_{L^1_\tau} \sum_{n_1,n_2\in\mathbb{Z}^2} \prod_{j=1}^{2} |\widehat{g}(n_j)|^2\\ &= \|\psi\|_{L^1_\tau} \Big(\sum_{n_1\in\mathbb{Z}^2} |\widehat{g}(n_1)|^2 \Big) \Big(\sum_{n_2\in\mathbb{Z}^2} |\widehat{g}(n_2)|^2 \Big)\\ &= \|\psi\|_{L^1_\tau}\ \|\widehat{g}\|_{\ell^2_n}^4\\ &\lesssim \|g\|_{L^2_x}^4. \qquad (7.1.4)\end{aligned}$$

So, it suffices to show

$$\| \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\tau + |n_1|^2 + |n_2|^2)| \, \|_{\ell^\infty_n L^\infty_\tau} \lesssim N^\varepsilon. \tag{7.1.5}$$

Let $\zeta := \tau + |n_1|^2 + |n_2|^2$ and $\varepsilon_0 > 0$ be a small quantity dependent on ε which we will specify later. By the triangle inequality, we have

$$\| \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\zeta)| \, \|_{\ell^\infty_n L^\infty_\tau} \leq \| \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\zeta)| \, \mathbb{1}_{|\zeta|\gtrsim N^{\varepsilon_0}} \|_{\ell^\infty_n L^\infty_\tau} + \| \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\zeta)| \, \mathbb{1}_{|\zeta|\ll N^{\varepsilon_0}} \|_{\ell^\infty_n L^\infty_\tau}$$

We bound each term individually. If $|\zeta| \gtrsim N^{\varepsilon_0}$, then since $\psi \in \mathcal{S}_\tau(\mathbb{R})$, it will decay faster than any negative power of $|\zeta|$, and so we can choose $k \geq \frac{2}{\varepsilon_0}$ to get

$$\| \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\zeta)| \, \mathbb{1}_{|\zeta|\gtrsim N^{\varepsilon_0}} \|_{\ell^\infty_n L^\infty_\tau} \lesssim \| \sum_{\substack{n_1\in\mathbb{Z}^2 \\ |n_1|\leq N}} \frac{1}{N^{k\varepsilon_0}} \|_{\ell^\infty_n L^\infty_\tau} \lesssim 1. \tag{7.1.6}$$

Otherwise, we have $|\zeta| \ll N^{\varepsilon_0}$. Fix τ and n. Notice that $|n_1|^2 + |n_2|^2 = -\tau + O(N^{\varepsilon_0})$. Since $n_2 = n - n_1$, we can write

$$|n_1|^2 + |n_2|^2 = |n_1|^2 + |n - n_1|^2 = 2|n_1|^2 - 2n_1 n + |n|^2 = \frac{1}{2}|2n_1 - n|^2 + \frac{1}{2}|n|^2,$$

so we have

$$R := |2n_1 - n|^2 = -2\tau - |n|^2 + O(N^{\varepsilon_0}).$$

This means that there are $O(N^{\varepsilon_0})$ many choices for $R \in \mathbb{Z}$. Moreover, since $|n_1|, |n_2| \leq N$, we have $|n| \lesssim N$, and $-\tau + O(N^{\varepsilon_0}) = |n_1|^2 + |n_2|^2 \lesssim N^2$, so $|\tau| \lesssim N^2$, meaning that we also have $|R| \lesssim N^2$.

Fix R satisfying the above. See that $|2n_1 - n|^2 = R$ if and only if

$$2n_1 - n \big| R \quad \text{in} \quad \mathbb{Z}[i] \cong \mathbb{Z}^2.$$

Thus, we apply Lemma 7.1.2 to find that there are $O(R^{\varepsilon_0}) = O(N^{2\varepsilon_0})$ choices for $2n_1 - n$ and consequently for n_1 since n is fixed. Finally, since there are $O(N^{\varepsilon_0})$ choices for R, we have in total $O(N^{3\varepsilon_0})$ many choices for (n_1, n_2) because n is fixed. Therefore, we have

$$\sup_{\tau,n} \sum_{\substack{n_1+n_2=n \\ |n_1|,|n_2|\leq N}} |\psi(\zeta)| \, \mathbb{1}_{|\zeta|\ll N^{\varepsilon_0}} \lesssim N^{3\varepsilon_0} \lesssim N^\varepsilon \tag{7.1.7}$$

provided that we choose $\varepsilon_0 \leq \frac{\varepsilon}{3}$. Then, (7.1.6) and (7.1.7) imply (7.1.5), which, together with (7.1.4) gives the desired result when substituted in (7.1.3). □

See also [16] for a proof of Proposition 7.1.1 using incidence geometry and a

discussion on the implicit constant in the inequality. By the transference principle (Lemma 5.2.1), we have the following corollary of Proposition 7.1.1:

Corollary 7.1.3. *Let $b > \frac{1}{2}$ and $u \in X^{0,b}$. Then, we have*

$$\|\eta(t)\, P_{\leq N} u\|_{L^4_{t,x}(\mathbb{R}_t \times \mathbb{T}^2_x)} \lesssim N^{\varepsilon} \|P_{\leq N} u\|_{X^{0,b}} \tag{7.1.8}$$

for all $\varepsilon > 0$.

Although the above is the desired Strichartz estimate in terms of the Fourier restriction norm, we are faced with a derivative loss in the form of an extra N^{ε} factor. That is, we have a weaker estimate than the corresponding Strichartz estimate in the one dimensional setting (6.1.5). As such, we cannot derive a non-linear estimate in the same way to conclude the contraction mapping argument, and the difficulty of the two dimensional case stems from the need to mitigate this derivative loss. In this aim, we derive a series of estimates to improve (7.1.8) qualitatively.

First, we seek improvement in terms of frequency localisation. Let Q be a cube in $\mathbb{R}^2$, and define the frequency projection operator P_Q by

$$\widehat{P_Q f} := \mathbb{1}_Q(n) \widehat{f}(n).$$

We have a second direct corollary of Proposition 7.1.1:

Corollary 7.1.4. *Let $\varepsilon > 0$ and $f \in L^2_x(\mathbb{T}^2)$. The following estimate holds*

$$\|\eta(t)\, S(t) P_Q f\|_{L^4_{t,x}(\mathbb{R}_t \times \mathbb{T}^2_x)} \lesssim |Q|^{\varepsilon} \|P_Q f\|_{L^2_x(\mathbb{T}^2)}. \tag{7.1.9}$$

Note that in (7.1.9) the bound does not depend on the centre of the cube Q, only its size. This means that Q may be centred arbitrarily far away from the origin while still retaining the same bound.

Proof (of Corollary 7.1.4). We do a direct computation. Let Q be a cube in $\mathbb{R}^2$ with centre n_0. We may assume that $n_0 \in \mathbb{Z}^2$, since otherwise we can run the same argument using a cube $\tilde{Q}$ of side-length, say, 4 times the side-length of Q, centred at the point $\tilde{n}_0 \in \mathbb{Z}^2$ nearest to $n_0 \in \mathbb{R}^2 \setminus \mathbb{Z}^2$. Consider

$$S(t) P_Q f(x) = \sum_{n \in \mathbb{Z}^2} e^{i(n \cdot x - t|n|^2)}\, \mathbb{1}_Q(n) \widehat{f}(n).$$

See that $n = k + n_0$ for some $k \in Q - n_0$, which is a cube centred at the origin. Then, $|n|^2 = |n_0|^2 + 2n_0 \cdot k + |k|^2$. So, we write the exponential in the above sum as

$$e^{i(n \cdot x - t|n|^2)} = (e^{i n_0 \cdot x}\, e^{-it|n_0|^2})\, e^{ik \cdot (x - 2n_0 t)}\, e^{-it|k|^2}.$$

Note that the first two factors of right-hand side above are independent of k. We

can thus write

$$\begin{aligned}
&\|\eta(t)\, S(t)P_Q f\|_{L^4_{t,x}} \\
&\quad = \||e^{in_0\cdot x}\, e^{-it|n_0|^2}|\, \eta(t) \sum_{k\in\mathbb{Z}^2} e^{ik\cdot(x-2n_0t)}\, e^{-it|k|^2}\, \mathbb{1}_{Q-n_0}(k)\widehat{f}(k+n_0)\|_{L^4_{t,x}} \\
&\quad = \|\eta(t) \sum_{k\in\mathbb{Z}^2} e^{ik\cdot(x-2n_0t)}\, e^{-it|k|^2}\, \mathbb{1}_{Q-n_0}(k)\widehat{f}(k+n_0)\|_{L^4_{t,x}}.
\end{aligned}$$

Let $y := x - 2n_0t$. Then, the above becomes

$$\begin{aligned}
\|\eta(t)\, S(t)P_Q f\|_{L^4_{t,x}} &= \|\eta(t) \sum_{k\in\mathbb{Z}^2} e^{i(k\cdot y - t|k|^2)}\, \mathbb{1}_{Q-n_0}(k)\widehat{f}(k+n_0)\|_{L^4_{t,x}} \\
&= \|\eta(t)\, e^{-in_0 y}\, S(t)P_{Q-n_0} f(y)\|_{L^4_{t,y}} \\
&= \|\eta(t)\, S(t)P_{Q-n_0} e^{-in_0 x} f(x)\|_{L^4_{t,x}} \\
&\lesssim |Q-n_0|^{\varepsilon}\|P_{Q-n_0} e^{-in_0 x} f\|_{L^2_x(\mathbb{T}^2)} \\
&= |Q|^{\varepsilon}\|P_Q f\|_{L^2_x(\mathbb{T}^2)},
\end{aligned}$$

by Proposition 7.1.1. □

The transference principle (Lemma 5.2.1) implies that we also have the following:

Corollary 7.1.5. *Let $b > \frac{1}{2}$ and $u \in X^{0,b}$. Then, we have*

$$\|\eta(t)\, P_Q u\|_{L^4_{t,x}(\mathbb{R}_t\times\mathbb{T}^2_x)} \lesssim |Q|^{\varepsilon}\|P_Q u\|_{X^{0,b}} \tag{7.1.10}$$

for all $\varepsilon > 0$.

The second improvement we seek is in regularity. So far, we are limited to $b > \frac{1}{2}$. However, to conduct a duality argument in proving the nonlinear estimate, we hope to pass to the space $X^{0,\frac{1}{2}-2\varepsilon}$. We achieve this by interpolating (7.1.10) against the low-regularity estimate below:

Lemma 7.1.6. *Let Q be a cube in $\mathbb{R}^2$. We have*

$$\|\eta(t)\, P_Q u\|_{L^4_{t,x}(\mathbb{R}_t\times\mathbb{T}^2_x)} \lesssim |Q|^{\frac{1}{4}}\|P_Q u\|_{X^{0,\frac{1}{4}}}. \tag{7.1.11}$$

Proof. We may translate Bernstein's inequality (2.1.1) to a result replacing $P_{\leq N}$ and N^d by P_Q and $|Q|$, respectively, using arguments similar to the proof of Corollary 7.1.4. Applying this result (in x), we have that

$$\|\eta(t)\, P_Q u\|_{L^4_{t,x}} \lesssim |Q|^{\frac{1}{2}-\frac{1}{4}}\, \|P_Q u\|_{L^4_t L^2_x}$$

for any t. The Sobolev inequality (2.0.1) (in t) gives

$$\begin{aligned}\|P_Q u\|_{L^4_t L^2_x} &\lesssim \|\langle \partial_t \rangle^{\frac{1}{4}} P_Q u\|_{L^2_{t,x}} \\ &= \|\langle \tau \rangle^{\frac{1}{4}} \widehat{P_Q u}\|_{\ell^2_n L^2_\tau} \\ &\lesssim \|\langle \tau + |n|^2 \rangle^{\frac{1}{4}} \widehat{P_Q u}\|_{\ell^2_n L^2_\tau} \\ &= \|P_Q u\|_{X^{0,\frac{1}{4}}}\end{aligned}$$

since $\sup_{n\in Q} |n|^2 \lesssim 1$ and any $n \notin Q$ does not contribute to the ℓ^2_n norm due to the support of $\widehat{P_Q u}$. Combining the two inequalities yields the desired estimate. □

Interpolating (7.1.10) and (7.1.11) using Riesz–Thorin interpolation (Lemma 2.0.2), we obtain

$$\|\eta(t)\, P_Q u\|_{L^4_{t,x}(\mathbb{R}_t \times \mathbb{T}^2_x)} \lesssim |Q|^{(1-\theta)\varepsilon_1 + \frac{\theta}{4}} \|P_Q u\|_{X^{0,(1-\theta)b+\frac{\theta}{4}}}$$

for $b > \frac{1}{2}$, $\varepsilon_1 > 0$, and $0 < \theta < 1$. Let $\varepsilon_0 := (1-\theta)\varepsilon_1 + \frac{\theta}{4}$. By choosing ε_1 and θ small enough, we can make ε_0 arbitrarily small, while ensuring that $(1-\theta)b + \frac{\theta}{4} < \frac{1}{2}$ by choosing b close enough to $\frac{1}{2}$. Thus, we have the following lemma:

Lemma 7.1.7. *Let $\varepsilon > 0$ and $u \in X^{0,\frac{1}{2}-2\varepsilon}$. Then,*

$$\|\eta(t)\, P_Q u\|_{L^4_{t,x}(\mathbb{R}_t \times \mathbb{T}^2_x)} \lesssim |Q|^{\varepsilon_0} \|P_Q u\|_{X^{0,\frac{1}{2}-2\varepsilon}} \tag{7.1.12}$$

for any $\varepsilon_0(\varepsilon) > 0$.

Although there is still a small derivative loss, the above lemma will be sufficient to prove the nonlinear estimate.

7.2 Contraction mapping argument

Recall the solution map

$$\Gamma_{u_0}(u) := \eta(t)\, S(t)u_0 \mp i\eta\left(\frac{t}{T}\right) \int_0^t S(t-t')|u|^2 u(t')\, dt'.$$

Since $[0,t] \subset [-1,1]$, we may write

$$\Gamma_{u_0}(u) = \eta(t)\, S(t)u_0 \mp i\eta\left(\frac{t}{T}\right) \int_0^t S(t-t')\, \eta(t')\, |u|^2 u(t')\, dt'.$$

So, to obtain Theorem 1.0.2, it suffices to show the following nonlinear estimate:

Proposition 7.2.1 (Nonlinear estimate in $\mathbb{T}^2$ [3])**.** *Let $s > 0$, and suppose $u_1, u_2, u_3 \in X^{s,\frac{1}{2}+\varepsilon}$. We have*

$$\|\eta(t)\, u_1 \overline{u_2} u_3\|_{X^{s,-\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}} \tag{7.2.1}$$

for all $0 < \varepsilon \leq \frac{1}{16}$.

Sadly, the nonlinear estimate on $\mathbb{T}^2$ does not hold when $s = 0$. The culprit is the derivative loss in the Strichartz estimate (7.1.12). This means that we fall just short of obtaining a local well-posedness result in the critical regularity setting. The local well-posedness of the cubic NLS on $L^2(\mathbb{T}^2)$ is still an open problem. We will discuss some related results in Chapter 8.

Proof (of Proposition 7.2.1). We begin the same way as in the proof of Proposition 6.2.1. See that, by duality we have

$$\begin{aligned}\|\eta(t)\, u_1\overline{u_2}u_3\|_{X^{s,-\frac{1}{2}+2\varepsilon}} &= \|\langle\nabla\rangle^s \eta(t)\, (u_1\overline{u_2}u_3)\|_{X^{0,-\frac{1}{2}+2\varepsilon}} \\ &= \sup_{\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \langle\nabla\rangle^s \eta(t)\, (u_1\overline{u_2}u_3)\overline{u_4}\, dx\, dt \right|.\end{aligned}$$

Fix u_4 with $\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} = 1$. We perform a dyadic decomposition of u_j for $j = 1, 2, 3, 4$. Write

$$u_j = \sum_{\substack{N_j \geq 1 \\ \text{dyadic}}} P_{N_j} u_j.$$

Then, we have

$$\begin{aligned}&\left| \iint_{\mathbb{R}\times\mathbb{T}^2} \langle\nabla\rangle^s \eta(t)\, (u_1\overline{u_2}u_3)\overline{u_4}\, dx\, dt \right| \\ &\qquad \leq \sum_{\substack{N_1,N_2,N_3,N_4 \geq 1 \\ \text{dyadic}}} \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \langle\nabla\rangle^s \eta(t)\, (P_{N_1}u_1\overline{P_{N_2}u_2}P_{N_3}u_3)\overline{P_{N_4}u_4}\, dx\, dt \right|.\end{aligned}$$

Notice that directly applying Hölder's inequality along with the Strichartz estimate (7.1.8) does not give the desired bound (7.2.1). In particular, (7.1.8) does not hold when $b \leq \frac{1}{2}$, meaning that we cannot evaluate the supremum as directly. More problematically, the derivative loss makes recovering the full norm from a frequency decomposition of the left-hand side of (7.2.1) impossible. On the other hand, the estimate (7.1.12) is enough to resolve these issues.

Fix some N_1, N_2, N_3, N_4. Let $N_{\max} := \max\{N_1, N_2, N_3, N_4\}$, and assume without loss of generality that $N_1 \geq N_2 \geq N_3$. Throughout this proof, when comparing the N_j to any quantity, we will be using the symbols $\sim$ and $\lesssim$ as outlined in Remark 2.1.1. See that

$$\begin{aligned}&\left| \iint_{\mathbb{R}\times\mathbb{T}^2} \langle\nabla\rangle^s \eta(t)\, (P_{N_1}u_1\overline{P_{N_2}u_2}P_{N_3}u_3)\overline{P_{N_4}u_4}\, dx\, dt \right| \\ &\qquad \lesssim N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^{4} P_{N_j}u_j^*\, dx\, dt \right|,\end{aligned}$$

where $P_{N_j} u_j^* = P_{N_j} u_j$ if j is odd and $P_{N_j} u_j^* = \overline{P_{N_j} u_j}$ if j is even. Moreover,

$$\begin{aligned}\Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx\, dt \Big| \\ = \Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \sum_{|n_j|\sim N_j} \widehat{u_1}(n_1)\, \overline{\widehat{u_2}(n_2)}\, \widehat{u_3}(n_3)\, \overline{\widehat{u_4}(n_4)}\, e^{ix\cdot(n_1-n_2+n_3-n_4)}\, dx\, dt \Big|,\end{aligned}$$

which vanishes except for when $n_1 - n_2 + n_3 - n_4 = 0$. Under this restriction, we have either $N_{\max} \sim N_1 \sim N_2$ or $N_{\max} \sim N_1 \sim N_4 \gg N_2 \sim N_3$. Thus, it suffices to show that

$$\sum_{\substack{N_1\sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} N_{\max}^s \Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx\, dt \Big| \lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}}, \quad \text{and} \tag{7.2.2}$$

$$\sum_{\substack{N_1\sim N_4, N_2, N_3 \geq 1 \\ \text{dyadic}}} N_{\max}^s \Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx\, dt \Big| \lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}}. \tag{7.2.3}$$

- **Case 1:** $N_{\max} \sim N_1 \sim N_2$.

Let η_j be cutoff functions such that $\prod_{j=1}^4 \eta_j = \eta$. Notice that by Hölder's inequality and the Strichartz estimate (7.1.12) we have

$$\begin{aligned}\Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx\, dt \Big| &\lesssim \prod_{j=1}^4 \|\eta_j(t)\, P_{N_j} u_j\|_{L^4_{t,x}} \\ &\lesssim \prod_{j=1}^4 N_j^{\varepsilon_0} \|P_{N_j} u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim \prod_{j=1}^4 N_j^{\varepsilon_0} \prod_{j=1}^3 \|u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}}\end{aligned}$$

since $\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} = 1$, for some $\varepsilon_0 > 0$ we will specify later. Thus,

$$\begin{aligned}\sum_{\substack{N_1\sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} N_{\max}^s \Big| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx\, dt \Big| \\ \lesssim \sum_{\substack{N_1\sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} N_{\max}^s \prod_{j=1}^4 N_j^{\varepsilon_0} \prod_{j=1}^3 \|u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Notice that $N_{\max}^s \lesssim N_1^{\frac{s}{2}} N_2^{\frac{s}{2}}$, and so

$$N_{\max}^s \prod_{j=1}^4 N_j^{\varepsilon_0} \lesssim N_1^{\frac{s}{2}+\varepsilon_0} N_2^{\frac{s}{2}+\varepsilon_0} N_3^{\varepsilon_0} N_4^{\varepsilon_0} \lesssim N_1^{\frac{s}{2}+2\varepsilon_0} N_2^{\frac{s}{2}+\varepsilon_0} N_3^{\varepsilon_0}.$$

Choosing $\varepsilon_0 > 0$ small enough such that $\frac{s}{2} + 3\varepsilon_0 \leq s$, we get

$$N_1^{\frac{s}{2}+2\varepsilon_0} N_2^{\frac{s}{2}+\varepsilon_0} N_3^{\varepsilon_0} \lesssim N_1^{s-\varepsilon_0} N_2^s N_3^s,$$

and since $N_1^{-\varepsilon_0} \sim N_{\max}^{-\varepsilon_0} \leq \prod_{j=1}^4 N_j^{-\frac{\varepsilon_0}{4}}$, we have

$$N_1^{s-\varepsilon_0} N_2^s N_3^s \leq N_1^s N_2^s N_3^s \prod_{j=1}^4 N_j^{-\frac{\varepsilon_0}{4}}.$$

As a result, we have that

$$\begin{aligned}
\sum_{\substack{N_1 \sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx \, dt \right| & \\
&\lesssim \sum_{\substack{N_1 \sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} \prod_{j=1}^4 N_j^{-\frac{\varepsilon_0}{4}} \prod_{j=1}^3 N_j^s \|u_j\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\
&\lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,\frac{1}{2}-2\varepsilon}} \sum_{\substack{N_1 \sim N_2, N_3, N_4 \geq 1 \\ \text{dyadic}}} \prod_{j=1}^4 N_j^{-\frac{\varepsilon_0}{4}} \\
&\lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}},
\end{aligned}$$

which is exactly (7.2.2).

- **Case 2:** $N_{\max} \sim N_1 \sim N_4 \gg N_2$.

Since we must eliminate the norm $\|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}$, we cannot have any derivative loss in the form of a N_4^s factor. This was not an issue in the previous case, but here, if we were to follow the same argument, we would not be able to create a $N_1^{-\varepsilon_0}$ factor to ensure summability in N_1 in the last step. Our remedy comes from the restriction $n_1 - n_2 + n_3 - n_4 = 0$, since it means that we have $n_4 = n_1 + O(N_2)$. That is, although N_4 is large, n_4 only varies a relatively small amount from n_1. The Strichartz estimate (7.1.12) allows us to take advantage of this localisation.

Consider the annuli $A_j := \{n_j \in \mathbb{Z}^2 : |n_j| \sim N_j\}$ for $j = 1, 4$. We may find finite collections of cubes Q_j^1 and Q_k^4 with $|Q_j^1| = |Q_k^4| \sim N_2^2$ for all j, k, such that

$$A_1 \subseteq \bigcup_j Q_j^1 \quad \text{and} \quad A_4 \subseteq \bigcup_k Q_k^4.$$

In particular, if we fix some n_1 and j such that $n_1 \in Q_j^1$, then there can only be $O(1)$ many choices for k such that $n_4 \in Q_k^4$ since $n_4 = n_1 + O(N_2)$. Figure 7.1 illustrates this situation.

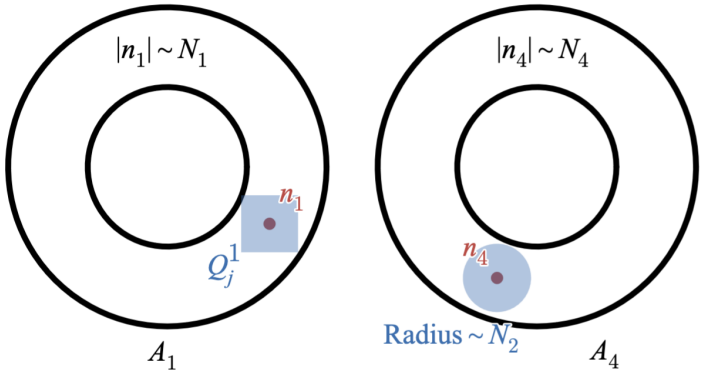


Figure 7.1: The annuli A_1, A_4, along with the locations of the frequencies n_1, n_4 for some j.

For fixed N_1, N_2, N_3, N_4, we can write

$$P_{N_1} u_1 = P_{Q_j^1} P_{N_1} u_1 \quad \text{and} \quad P_{N_4} u_4 = P_{Q_k^4} P_{N_4} u_4$$

for some j, k, and so we have

$$\begin{aligned}
&N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx \, dt \right| \\
&\le N_{\max}^s \sum_j \sum_k \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \, (P_{Q_j^1} P_{N_1} u_1) \, \overline{(P_{Q_k^4} P_{N_4} u_4)} \, \overline{P_{N_2} u_2} \, P_{N_3} u_3 \, dx \, dt \right| \\
&\le N_{\max}^s \sum_j \sum_k \|P_{Q_j^1} P_{N_1} u_1\|_{L^4_{t,x}} \|P_{Q_k^4} P_{N_4} u_4\|_{L^4_{t,x}} \prod_{\ell=2}^3 \|P_{N_\ell} u_\ell\|_{L^4_{t,x}} \\
&\lesssim N_{\max}^s \sum_{j,\, k(j)} N_2^{\varepsilon_0} \|P_{Q_j^1} P_{N_1} u_1\|_{X^{0,\frac{1}{2}-2\varepsilon}} \, N_2^{\varepsilon_0} \|P_{Q_k^4} P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \prod_{\ell=2}^3 N_\ell^{\varepsilon_0} \|P_{N_\ell} u_\ell\|_{X^{0,\frac{1}{2}-2\varepsilon}}
\end{aligned}$$

by Hölder's inequality and the Strichartz estimate (7.1.12), since $|Q_j^1| = |Q_k^4| \sim N_2^2$. Choosing $\varepsilon_0 > 0$ such that $4\varepsilon_0 \le s$, we get $N_2^{3\varepsilon_0} N_3^{\varepsilon_0} \le N_2^{s-\varepsilon_0} N_3^{s-\varepsilon_0}$. Further note that $N_{\max}^s \lesssim N_1^s$. Thus, we have

$$\begin{aligned}
&N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx \, dt \right| \\
&\lesssim N_2^{-\varepsilon_0} N_3^{-\varepsilon_0} \sum_j \sum_k \|P_{Q_j^1} P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{Q_k^4} P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \prod_{\ell=2}^3 \|u_\ell\|_{X^{s,\frac{1}{2}-2\varepsilon}}
\end{aligned}$$

since $N_\ell^s \|P_{N_\ell} u_\ell\|_{X^{0,\frac{1}{2}-2\varepsilon}} \lesssim \|P_{N_\ell} u_\ell\|_{X^{s,\frac{1}{2}-2\varepsilon}}$ for $\ell = 1,2,3$. Applying Cauchy–Schwarz in the sum over j, we get

$$\begin{aligned}\sum_j \sum_k & \|P_{Q_j^1} P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{Q_k^4} P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\leq \left(\sum_j \|P_{Q_j^1} P_{N_1} u_1\|^2_{X^{s,\frac{1}{2}-2\varepsilon}}\right)^{\frac{1}{2}} \left(\sum_j \sum_k \|P_{Q_k^4} P_{N_4} u_4\|^2_{X^{0,\frac{1}{2}-2\varepsilon}}\right)^{\frac{1}{2}} \\ &\lesssim \|P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \left(\sum_j \sum_k \|P_{Q_k^4} P_{N_4} u_4\|^2_{X^{0,\frac{1}{2}-2\varepsilon}}\right)^{\frac{1}{2}}.\end{aligned}$$

We may write the double sum on the right as a sum over all k with $j = j(k)$ dependent on k instead. Since for a given k, there are $O(1)$ choices for j, we have

$$\begin{aligned}\sum_j \sum_k & \|P_{Q_j^1} P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{Q_k^4} P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim \|P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \left(\sum_k \sum_{j(k)} \|P_{Q_k^4} P_{N_4} u_4\|^2_{X^{0,\frac{1}{2}-2\varepsilon}}\right)^{\frac{1}{2}} \\ &\lesssim \|P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}},\end{aligned}$$

meaning that

$$\begin{aligned}N_{\max}^s & \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx \, dt \right| \\ &\lesssim N_2^{-\varepsilon_0} N_3^{-\varepsilon_0} \|P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \prod_{\ell=2}^3 \|u_\ell\|_{X^{s,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Similar to the previous case, we use the $N_\ell^{-\varepsilon_0}$ to sum over N_2, N_3, and as such we get

$$\begin{aligned}\sum_{\substack{N_1 \sim N_4, N_2, N_3 \geq 1 \\ \text{dyadic}}} & N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^4 P_{N_j} u_j^* \, dx \, dt \right| \\ &\lesssim \prod_{\ell=2}^3 \|u_\ell\|_{X^{s,\frac{1}{2}-2\varepsilon}} \sum_{\substack{N_1 \sim N_4 \geq 1 \\ \text{dyadic}}} \|P_{N_1} u_1\|_{X^{s,\frac{1}{2}-2\varepsilon}} \|P_{N_4} u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Since $N_1 \sim N_4$, we may write $N_4 = 2^m N_1$ with $|m| \leq 3$, thus given N_1, there are $O(1)$ choices for N_4. As such, by an application of Cauchy–Schwarz similar to

earlier, we receive

$$\sum_{\substack{N_1 \sim N_4, N_2, N_3 \geq 1 \\ \text{dyadic}}} N_{\max}^s \left| \iint_{\mathbb{R}\times\mathbb{T}^2} \eta(t) \prod_{j=1}^{4} P_{N_j} u_j^* \, dx \, dt \right|$$

$$\lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,\frac{1}{2}-2\varepsilon}} \, \|u_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,\frac{1}{2}+\varepsilon}},$$

which is exactly (7.2.3). □

Using the nonlinear estimate (7.2.1), one may conclude the contraction mapping argument in the same way as in Chapter 6.2, this time with $s > 0$, and receive that the cubic NLS is locally well-posed in $H^s(\mathbb{T}^2)$. This proves Theorem 1.0.2.

Chapter 8

Conclusion

We conclude by remarking on results following Bourgain's 1993 paper [3]. We first consider results regarding the one-dimensional cubic NLS. In Chapter 6 we proved the local well-posedness of the cubic NLS on $H^s(\mathbb{T})$ for $s \geq 0$, which was originally shown in [3]. This result is sharp, and solutions do not exist when $s \in (-\frac{1}{8}, 0)$. On the other hand, in this low regularity setting, one may renormalise the cubic NLS via a gauge transform to obtain the Wick ordered cubic NLS whose solutions exist in local time. These existence and non-existence results can be found in [12]. Proving local well-posedness for the Wick ordered cubic NLS in $H^s(\mathbb{T})$ for $s \in (-\frac{1}{8}, 0)$ is an open problem. In contrast, the cubic NLS is globally well-posed in $H^s(\mathbb{R})$ for any $s > -\frac{1}{2}$, which was proven with non-harmonic analytic methods, using complete integrability in [14]. For critical and supercritical regularity $s \leq -\frac{1}{2}$ when $d = 1$ and for $s < 0$ when $d \geq 2$, the cubic NLS is ill-posed on $H^s(M^d)$ with $M = \mathbb{R}$ or $\mathbb{T}$ since the solution map is discontinuous everywhere in $H^s(M^d)$, see [25].

In two dimensions, we showed in Chapter 4 that the cubic NLS in $L^2(\mathbb{R}^2)$ is locally well-posed in the L^4-critical sense and globally well-posed for small initial data. Global well-posedness of the cubic NLS in $L^2(\mathbb{R}^2)$ for arbitrary initial data was shown in [7]. This is the critical regularity case $s = s_c = 0$. Combined with the supercritical ill-posedness result above in [25] (see also [22] and the preprint [4]), this provides a full scope of results concerning the cubic NLS on $\mathbb{R}^2$. In Chapter 7, we proved Bourgain's [3] result from 1993 that the cubic NLS is locally well-posed in $H^s(\mathbb{T}^2)$ for $s > 0$. The problem of proving local well-posedness of the cubic NLS on $L^2(\mathbb{T}^2)$ is still open. Global well-posedness of the cubic NLS on $H^s(\mathbb{T}^2)$ in the subcritical setting $s > 0$ was proven for small initial data recently in 2024 by Herr–Kwak [16] by improving on [3] using incidence geometry, and they further relaxed the small initial data requirement to obtain general well-posedness for arbitrarily large initial data for the defocusing regime and for data below the ground state threshold in the focusing regime, in the 2025 preprint [15].

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